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MATHEMATICAL REVIEWS

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HISTORY

Adamo, Marco. La cultura geometrica dei popoli antichi. La geometria particolarmente rappresentativa presso i popoli della Mesopotamia. Rend. Sem. Fac. Sci. Univ. Cagliari 22 (1952), supplemento, 13–88 (1953).

Frajese, Attilio. La scoperta dell'incommensurabile nel dialogo "Menone". Boll. Un. Mat. Ital. (3) 9, 74–80 (1954).

Neugebauer, O. On the "Hippopede" of Eudoxus. Scripta Math. 19 (1953), 225–229 (1954).

Milankovitch, M. Ueber die Ausrechnung des Claudius Ptolemaus der Zahl π. Srpska Akad. Nauka. Zbornik Radova 35, Mat. Inst. 3, 11–14 (1953). (Serbo-Croatian. German summary)

Hermelink, Heinrich. Ein bisher übersehener Fehler in einem Beweis des Archimedes. Arch. Internat. Hist. Sci. (N.S.) 6, 430–433 (1953).

Hofmann, Jos. E. Über eine altindische Berechnung von π und ihre allgemeine Bedeutung. Math.-Phys. Semesterber. 3, 193–206 (1953).

This is a discussion of three papers by Rajagopal and Aiyar [Scripta Math. 15, 201–209 (1949); 17, 65–74 (1951); 18, 25–30 (1952); these Rev. 11, 572; 13, 1; 14, 121]. The series $\frac{1}{4}\pi=\sum_{k=0}^{\infty}(-1)^k(2k+1)^{-1}$ is not well-adapted to the computation of π. Let $S_n=\sum_{k=0}^{n-1}(-1)^k(2k+1)^{-1}$ and let T_n be one of the following correction-terms:

$$\frac{(-1)^n}{4n},\quad (-1)^n\frac{n}{4n^2+1},\quad (-1)^n\frac{n^2+1}{n(4n^2+5)}.$$

Using these estimates, the Hindus were able to derive new series more suitable for computation. The author discusses these estimates, the Hindu approximation methods, and the use of some of the methods on power series as well as on numerical series. *E. B. Allen* (Troy, N. Y.).

Luce, Jean-H. Géométrie de la perspective à l'époque de Vitruve. Rev. Hist. Sci. Appl. 6 (1953), 308–321 (1954).

Itard, Jean. Sur l'histoire des fractions continues. Rev. Gén. Sci. Pures Appl. 61, 5–18 (1954).

Boyer, Carl B. Early contributions to analytic geometry. Scripta Math. 19 (1953), ~~230–238~~ (1954).
97–108, 230–238

Adamo, Marco. Le fonti della storia della geometria particolarmente rappresentativa. Rend. Sem. Fac. Sci. Univ. Cagliari 22 (1952), supplemento, 1–12 (1953).

Ugo Amaldi. Accad. Naz. dei XL. Annuario Generale 1953, 209–213 (1 plate) (1954).
A list of publications is included.

Bernoulli, Daniel. Exposition of a new theory on the measurement of risk. Econometrica 22, 23–36 (1954).

Translation by L. Sommer of Bernoulli's "Specimen theoriae novae de mensura sortis" [Comment. Acad. Sci. Imp. Petropol. 5 (1730–1731), 175–192 (1738)].

***Bianchi, Luigi. Opere. Vol. II. Applicabilità e problemi di deformazione.** A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese della Casa Editrice Perrella, Roma, 1953. 337 pp. 3000 Lire.

For vol. I see these Rev. 14, 524; 15, 276. This volume contains Bianchi's papers on the subject of the subtitle. There is an introductory essay by R. Calapso on Bianchi's work in this field.

Enrico Bompiani. Accad. Naz. dei XL. Annuario Generale 1953, 297–309 (1 plate) (1954).
A list of publications is included.

Magnus, Wilhelm, und Moufang, Ruth. Max Dehn zum Gedächtnis. Math. Ann. 127, 215–227 (1954).

Hagstroem, K.-G. Gustaf Eneström. Nordisk Mat. Tidskr. 1, 145–155, 182 (1953). (Swedish. English summary)

Gillings, R. J. The so-called Euler-Diderot incident. Amer. Math. Monthly 61, 77–80 (1954).

Gaston Maurice Julia. Accad. Naz. dei XL. Annuario Generale 1953, 455–457 (1 plate) (1954).

Raševskiĭ, P. K. Obituary: Veniamin Fëdorovič Kagan. Uspehi Matem. Nauk (N.S.) 8, no. 5(57), 131–138 (1 plate) (1953). (Russian)
A list of Kagan's published papers is included.

Agostini, Amedeo. Quattro lettere inedite de Leibniz e una lettera di G. Grandi. Arch. Internat. Hist. Sci. (N.S.) 6, 434–443 (1953).

Rescher, Nicholas. Leibniz's interpretation of his logical calculi. J. Symbolic Logic 19, 1–13 (1954).

Signorini, A. Leonardo da Vinci e la meccanica. Scientia (6) 88, 1–10 (1953).

***Luzin, N. N. Sobranie sočinenii. Tom I. Metričeskaya teoriya funkcii i teoriya funkcii kompleksnogo peremennogo.** [**Collected works. Vol. I. Metric theory of functions and theory of functions of a complex variable.**] Izdat. Akad. Nauk SSSR, Moscow, 1953. 400 pp. (2 plates). 22 rubles.

All papers not originally in Russian have been translated into that language.

✱Mercalov, N. I. Izbrannye trudy. Tom I. Prikladnaya mehanika. [Selected works. Vol. I. Applied mechanics.] Gosudarstv. Naučno-Tehn. Izdat. Mašinostroit. Lit., Moscow, 1952. 368 pp. (2 plates). 10 rubles.

This volume contains Mercalov's lectures on "The general theory of mechanisms" (1904) and on "The general theory of machines" (1914).

Rodero, Julián. Mercator. Gaceta Mat. (1) **5**, 143–146 (1 plate) (1953). (Spanish)

✱Meščerskiĭ, I. V. Raboty po mehanike tel peremennoĭ massy. [Works on the mechanics of bodies of variable mass.] 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1952. 280 pp. 6 rubles.

The first edition was published in 1949. In addition to Meščerskiĭ's works on the subject of the title there is also an essay by Kosmodem'yanskiĭ on his work.

Quade, W. Obituary: Conrad Müller. Jber. Deutsch. Math. Verein. **57**, Abt. 1, 1–5 (1954).

Andrade, E. N. da C. Newton. Considérations sur l'homme et son oeuvre. Rev. Hist. Sci. Appl. **6** (1953), 289–307 (1954).

Mauro Picone. Accad. Naz. dei XL. Annuario Generale **1953**, 357–370 (1 plate) (1954).

A list of publications is included.

Humbert, Pierre. L'oeuvre mathématique d'Henri Pitot. Rev. Hist. Sci. Appl. **6** (1953), 322–328 (1954).

Tonolo, Angelo. Commemorazione di Gregorio Ricci-Curbastro nel primo centenario della nascita. Rend. Sem. Mat. Univ. Padova **23**, 1–24 (1 plate) (1954).

Francesco Severi. Accad. Naz. dei XL. Annuario Generale **1953**, 13–30 (1 plate) (1954).

A list of Severi's publications up to 1950 is included.

Carlo Somigliana. Accad. Naz. dei XL. Annuario Generale **1953**, 9–11 (1 plate) (1954).

Obituary: Gyula Szőkefalvi-Nagy. Acta Sci. Math. Szeged **15**, 97–98 (1954).

Carruccio, Ettore. Giovanni Vacca, matematico, storico e filosofo della scienza. Boll. Un. Mat. Ital. (3) **8**, 448–456 (1953).

Tenca, Luigi. Sulle vele secondarie di Vincenzio Viviani. Boll. Un. Mat. Ital. (3) **8**, 456–459 (1953).

FOUNDATIONS

Have

✱Fitch, Frederic Brenton. Symbolic logic, an introduction. The Ronald Press Company, New York, 1952. x+238 pp. $4.50.

Der vom Verf. gefundene Logikkalkül [vgl. J. Symbolic Logic **13**, 95–106 (1948); **14**, 209–218 (1950); diese Rev. **9**, 559; **12**, 2] wird in verbesserter Form ausführlich dargestellt, so dass das Buch zugleich als Einführung in die mathematische Logik dienen kann. Gegenüber den willkürlichen Einschränkungen des Komprehensionsaxioms, die an dem widerspruchsvollen Idealkalkül [vgl. Hermes und Scholz, Enzyk. Math. Wiss II, 1. Bd. I, Teubner, Leipzig, 1952] meist vorgenommen werden, ohne dass dadurch die Widerspruchsfreiheit beweisbar wird, ist die Fitch'sche Logik (1) finit als widerspruchsfrei beweisbar, (2) ausreichend zur Begründung einer Analysis [vgl. hierzu die Arbeiten des Verf. in J. Symbolic Logic, **14**, 9–15 (1949); **15**, 17–24 (1950); **16**, 121–124 (1951); diese Rev. **10**, 669; **12**, 2; **13**, 4]. Die verzweigte Typenlogik erfüllt zwar auch (1) und (2), aber die grosse Leistung des Verf. liegt in der Entdeckung, durch welche einfachen Einschränkungen der logischen Regeln die Erweiterung des Prädikatenkalküls durch das uneingeschränkte Komprehensionsaxiom widerspruchsfrei wird.

Schon innerhalb der positiven Logik wird die Behandlung der Implikation durch die Zulassung hypothetischer Unterbeweise wesentlich verbessert. Ein Beweis, der von der Hypothese p aus zu q führt, hat als unmittelbare Konsequenz $p\supset q$. Durch diese Methode der Unterbeweise wird die Deduktion "natürlicher". Es wird die folgende Einschränkung aufgestellt, die für die Widerspruchsfreiheit entscheidend ist; zu einem Beweis von p darf kein Unterbeweis benutzt werden, der von p allein auf eine von p verschiedene Aussage q schliesst. D.h. von $p\supset q$ und $(p\supset q)\supset p$ darf nicht auf p geschlossen werden. Es erscheint plausibel, dass dieser spezielle Fall der modus ponens zum Aufbau der Mathematik entbehrlich ist. Abweichend vom üblichen ist auch die Behandlung der Negation $\sim$. Es fehlt das tertium non datur. Die Fitch'sche Logik enthält aber die de Morgan'schen Regeln—auch für die Quantoren—und die Äquivalenz von $\sim\sim p$ mit p, dagegen keine Äquivalenz für $\sim(p\supset q)$. Mit $\wedge$ statt $q\&\sim q$ wird $\wedge\supset p$ beibehalten, der indirekte Schluss $(p\supset\wedge)\supset\sim p$ aber zu $(p\vee\sim p)\&(p\supset\wedge)\supset\sim p$ abgeschwächt. Auch diese Abschwächungen beeinträchtigen die deduktive Leistung des Kalküls nur wenig.

Gegenüber der üblichen Vernachlässigung der Modallogik ist hervorzuheben, dass Verf. dieser Gebiet gleichwertig mit in Betracht zieht. Die benutzten Axiome sind (ohne Verwendung von Unterbeweisen) für die "Notwendigkeit" $\Box$: (1) $\Box p\supset p$, (2) ist $p_1\&p_2\&\cdots\&p_n\supset p$ ableitbar, dann $\Box p_1\&\Box p_2\&\cdots\&\Box p_n\supset\Box p$. Hinzugenommen wird (3) $\Box p\supset\Box\Box p$, dagegen für die "Möglichkeit" $\Diamond$ nicht $\Diamond\Diamond p\supset\Diamond p$. Wegen des fehlenden tertium non datur folgt dies nicht aus den geforderten Äquivalenzen von $\Diamond p$ mit $\sim\Box\sim p$ und von $\sim\Diamond p$ mit $\Box\sim p$. Es ist $\Box p\&q\supset\Box p\&\Box q$ ableitbar. Verf. hat $\Diamond p\vee\Diamond q\supset\Diamond p\vee q$, $(x)\Box\phi x\supset\Box(x)\phi x$ und $\Diamond(\exists x)\phi x\supset(\exists x)\Diamond\phi x$ nicht in seine Logik aufgenommen, die Hinzunahme dieser Implikationen wäre aber widerspruchsfrei. Die Identität wird mit den üblichen Axiomen $a=a$ und $a=b\&\phi a\supset\phi b$ eingeführt und $a=b\vee\sim a=b$ hinzugefügt. Der Kalkül ist so angelegt, dass alle Regeln entweder Regeln zur Einführung oder zur Beseitigung einer logischen Konstanten sind. Die Widerspruchsfreiheit ergibt sich daher aus einem Fundamentaltheorem, das dem Gentzen'schen Hauptsatz analog ist: zu jeder ableitbaren Aussage gibt es einen Beweis, der—ausser in den Unterbeweisen—nur Einführungsregeln benutzt. Die Behandlung der Quantoren im Widerspruchsfreiheitbeweis geschieht durch Verwendung unendlicher Konjunktionen und Disjunktionen. Der Widerspruchfreiheitbeweis bleibt aber finit.

Einige Anhänge, z.B. über die Kombinatorische Logik, bereichen das Buch, dem der Ref.—der persönlich die verzweigte Typenlogik der dargestellten Logik vorzieht—grösste Beachtung wünscht, um die Forderung, zur Grundlegung der Mathematik nur Kalküle zu verwenden, die finit

als widerspruchsfrei beweisbar sind, allmählich zu einer Selbstverständlichkeit werden zu lassen. Die Entschuldigung, solche Kalküle gäbe es nicht, kann nach diesem Buch niemand mehr gelten machen. *P. Lorensen.*

Fine, N. J. Proof of a conjecture of Goodman. J. Symbolic Logic **19**, 41–44 (1954).

Suppose we have a finite set S in which an order relation $<$, a relation of betweenness B and a relation for "matching" M are given, such that B implies M, M is reflexive and symmetric, whereas from $M(x, y)$, $x \leqq u$, $u \leqq v$, $v \leqq y$, it follows that $M(u, v)$. Moreover, let $S(x, y)$ be one more than the number of elements strictly between x and y, and let us write $J(b, a)$ whenever not $M(a, b)$ though there is a c such that $M(a, c)$ and $B(c, b)$. It is proved that the following conditions are equivalent: (i) if $M(x, y)$ and not $M(u, v)$, then $S(x, y) < S(u, v)$; (ii) if $B(x, y)$, $M(x, a)$, and $M(x, b)$, then either $a = b$, or $M(y, a)$, or $M(y, b)$; (iii) J is a symmetric relation; (iv) for some natural number k, we have $M(x, y)$ if and only if $S(x, y) \leqq k$.
E. W. Beth (Amsterdam).

Šanin, N. A. On some operations on logico-arithmetic formulas. Doklady Akad. Nauk SSSR (N.S.) **93**, 779–782 (1953). (Russian)

This is related to recent work on realizability, constructible falsity, etc., by S. C. Kleene and D. Nelson [see, e.g., Kleene, Introduction to metamathematics, Van Nostrand, New York, 1952, pp. 501 ff.; these Rev. **14**, 525]. The author characterizes two operations Δ and ∇ by inductive specifications of which the following are typical clauses:

$$\begin{array}{ll} \Delta(T=S).\equiv.T=S, & \nabla(T=S).\equiv.\urcorner(T=S) \\ \Delta(P\&Q).\equiv.\Delta P\&\Delta Q, & \nabla(P\&Q).\equiv.\nabla P\vee\nabla Q \\ \Delta(P\supset Q).\equiv.\nabla P\vee\Delta Q, & \nabla(P\supset Q).\equiv.\Delta P\&\nabla Q \\ \Delta(\forall xP).\equiv.\forall x(\Delta P), & \nabla(\forall xP).\equiv.(\exists x)\nabla P \end{array}$$

where T and S are terms, P and Q are formulas, and '$\equiv$' indicates identity by definition. He states various theorems concerning these operations; typical of these is the statement that $\Delta R \supset R$, $\nabla R \supset \urcorner R$ hold in the intuitionistic calculus, while the converses hold in the classical calculus but are not realizable. *H. B. Curry* (State College, Pa.).

L'Abbé, Maurice. Systems of transfinite types involving λ-conversion. J. Symbolic Logic **18**, 209–224 (1953).

Construction of formal systems S_a involving transfinite types is carried out on the basis of a system S_0 described by Church. Essential steps in the construction: (i) introduction of a set K^{a+1} of type symbols containing all ordinals $u \leqq a+1$, and containing (xy) whenever it contains x and y; (ii) introduction of variables, constants and well formed expressions A_x having the type indicated by the subscript for each x in K^{a+1}; following Kemeny and Henkin, the intended interpretation is described by associating to each x in K^{a+1} a domain D_x of values for A_x such that: D_0 contains the two truth values, D_1 is the set of all natural numbers, $D_{(xy)}$ is the set of all functions defined over D_y with values in D_x, and, for any ordinal $u \geqq 1$, D_u is the union of all domains D_x such that all ordinals in x are smaller than u; (iii) introduction of axioms and rules of inference; this step is carried out in detail for S_2, which is studied with the following results: (iv) existence proof for functions defined over D_1 by primitive recursion and with values in D_x (x in K^3); (v) proof of relative consistency for S_2 with respect to Zermelo set theory; (vi) consistency proof for S_0 within S_2, which shows that an objection to certain other formal systems with transfinite types, stated by Turing, has been successfully avoided. *E. W. Beth.*

McNaughton, Robert. Axiomatic systems, conceptual schemes, and the consistency of mathematical theories. Philos. Sci. **21**, 44–53 (1954).

The author defends a position related to ideas set forth, among others, by Bernays and by the reviewer, and for which he rightly refers to Descartes. If we reject the formalist doctrine according to which there is no mathematical wisdom beyond the fact that certain statements are theorems in certain axiomatic systems, it does not follow that we are committed to a platonistic realism or to Brouwer's intuitionism. It is still possible to hold that there is a notion of mathematical validity independent of axiomatic systems; this notion of validity is defined by reference to a conceptual scheme, which is characterised as a sort of intuitive model. For instance, our intuitive conception of the universe consisting of the entities $0, 1, 2, \cdots$ is the conceptual scheme of arithmetic. In support of this doctrine, it is argued that the only way to judge the consistency of a mathematical theory is by referring to a conceptual scheme. However, this appeal to a conceptual scheme can never justify a dogmatic belief in the consistency of a given theory; for we can never be absolutely certain that our intuition is not playing tricks on us. *E. W. Beth* (Amsterdam).

Suetuna, Zyoiti. Über die Grundlagen der Mathematik. III. Proc. Japan Acad. **29**, 91–95 (1953).

[For parts I–II see J. Math. Soc. Japan **3**, 59–68 (1951); Proc. Japan Acad. **27**, 389–392 (1951); these Rev. **13**, 310, 898.] From the author's philosophical point of view a set must be defined as a totality of elements which is constituted by an intuition brought about by action. He argues that Fraenkel's axioms of Ersetzung, Aussonderung and of the power set are not generally valid for this notion of a set; however, the latter is valid for denumerable sets.
A. Heyting (Amsterdam).

Wittenberg, Alexandre. Über adäquate Problemstellung in der mathematischen Grundlagenforschung. Dialectica **7**, 232–254 (1953).

From the author's point of view, each of the current opinions about the foundations of mathematics requires certain dogmatic presuppositions about a certain reality, which is either that of mathematical objects or that of deducible formulas in a formal system. As a way out he proposes an epistemological investigation of concepts which are not supposed beforehand to correspond to some reality.
A. Heyting (Amsterdam).

Brouwer, L. E. J. Points and spaces. Canadian J. Math. **6**, 1–17 (1 plate) (1954).

In the first part of this paper the intuitionist point of view on mathematics is exposed and the fundamental notions of intuitionistic mathematics are introduced. In the second part the proof of the fundamental theorems of spread theory [first formulated by the author, Akad. Wetensch. Amsterdam. Proc. **27**, 189–193, 644–646 (1924)] is given in a particularly clear form by means of an elaborate terminology. The corollary on the continuity of functions is proved in the more general and very useful form: Every full mapping of a located compact topological space onto another located compact topological space is uniformly continuous (located = katalogisiert). *A. Heyting.*

*Gonseth, Ferdinand. **La géométrie et le problème de l'espace. I. La doctrine préalable.** Editions du Griffon, Neuchâtel, 1945. 69 pp.

*Gonseth, Ferdinand. **La géométrie et le problème de l'espace. II. Les trois aspects de la géométrie.** Editions du Griffon, Neuchâtel, 1946. viii+90 pp.

*Gonseth, Ferdinand. **La géométrie et le problème de l'espace. III. L'édification axiomatique.** Editions du Griffon, Neuchâtel, 1947. 110 pp.

*Gonseth, Ferdinand. **La géométrie et le problème de l'espace. IV. La synthèse dialectique.** Editions du Griffon, Neuchâtel, 1949. 80 pp.

*Gonseth, Ferdinand. **La géométrie et le problème de l'espace. V. Les géométries non euclidiennes.** Editions du Griffon, Neuchâtel, 1952. 110 pp.

This is an elaborate study of the different phases through which geometrical knowledge passes on its way from intuitive conception to rational knowledge. In Chapter I the author exposes the underlying philosophical attitude of "idoneism". In Chapter II he studies the intuitive, the experimental and the theoretical aspects of geometry on the level of every-day life and the relations between them. In Chapter III the process of axiomatization is described in detail and carried through to a certain extent. In Chapter IV a synthesis is accomplished between the four aspects so far considered. In Chapter V the Lobachewsky geometry is introduced by means of the Poincaré model and its independent axiomatization is sketched. Stress is laid on the consequences which the possibility of non-euclidean geometry involves for the fundamental notions such as intuition, axiomatization and reality. A sixth chapter will contain the author's solution of the "space problem".

A. Heyting (Amsterdam).

*Vacca, Giovanni. **La costante di Eulero e l'aritmetica analitica.** Società Italiana per il Progresso delle Scienze, XLII riunione, Roma, 1949, Relazioni, Vol. primo, pp. 177–180. Società Italiana per il Progresso delle Scienze, Roma, 1951.

This consists of a number of sketchy remarks concerning the basic nature of mathematics. For example, the author comments that irrationality proofs usually arise in connection with some geometric interpretation. Again he states that mathematics is not a hypothetico-deductive science in the sense of Russell, but is concerned with the derivation of absolute truths. The only information about Euler's constant is a statement that it arises from a geometric problem having some analogies with problems leading to π and log 2.

H. B. Curry (State College, Pa.).

*De Finetti, Bruno. **La "logica del plausibile" secondo la concezione di Polya.** Società Italiana per il Progresso delle Scienze, XLII riunione, Roma, 1949, Relazioni, Vol. primo, pp. 227–236. Società Italiana per il Progresso delle Scienze, Roma, 1951.

Discussion of an example from the axiomatic theory of probability to illustrate Polya's conception of a logic of plausible inference.

E. W. Beth (Amsterdam).

Kamke, E. Werden und Sicherheit mathematischer Erkenntnis. Jber. Deutsch. Math. Verein. **57**, Abt. 1, 6–20 (1954).

ALGEBRA

*Mostowski, Andrzej, and Stark, Marceli. **Algebra wyższa. Część pierwsza.** [Higher algebra. First part.] Biblioteka Matematyczna, Tom I. Polskie Towarzystwo Matematyczne, Warszawa, 1953. vii+308 pp.

This is primarily a university textbook beginning with the material for first year students. The style is clear, proofs given in great detail and the didactic aspect of the presentation receives considerable attention. In line with this the presentation in the first half of the book follows the classical pattern and the more abstract methods are postponed to the second half. Content: Chapter I, Introduction (sets, functions, natural numbers, the principle of induction); II, Combinatorics; III, Complex numbers; IV, Determinants; V, Vector spaces and linear equations; VI, Linear transformations; VII, Linear and quadratic forms; VIII, Geometric theory of linear transformations and Hermitian forms. The book contains many examples and problems. The second volume is to treat more advanced and abstract parts of theory.

A. Zygmund.

Sierpiński, W. Sur un problème concernant un réseau à 36 points. Ann. Soc. Polon. Math. **24** (1951), no. 2, 173–174 (1954).

The following problem is due to Zarankiewicz. Let $n>3$. Denote by R_n the system of n^2 points situated on n horizontal and vertical lines (i.e., an n by n matrix). Determine the smallest integer $k(n)$ so that any subset of R_n having $k(n)$ elements contains 9 points situated on three horizontal and three vertical lines. It is easy to see that $k(4)=14$, $k(5)=21$. The author proves that $k(6)=27$.

P. Erdös.

Wallace, Andrew H. Generalised Young tableaux. Proc. Edinburgh Math. Soc. (2) **9**, 35–43 (1953).

Let (μ): $\mu_1+\cdots+\mu_r$ $(\mu_1\geqq\cdots\geqq\mu_r>0)$ be a partition of $n+m$, let (ρ): $\rho_1+\cdots+\rho_r$ $(\rho_1\geqq\cdots\geqq\rho_r\geqq 0)$ be a partition of m, and let (λ): $\lambda_1+\cdots+\lambda_r$ be the partition of n defined by $\lambda_i=\mu_i-\rho_i$. We suppose also (μ) and (ρ) are chosen so that $\lambda_1\geqq\cdots\geqq\lambda_r>0$. Let T_μ, T_ρ denote the Young tableaux defined by (μ) and (ρ) and denote by $T_{\mu-\rho}$ the tableaux obtained by deleting that part of T_μ which is covered by T_ρ when the two are placed with upper left hand corners coincident. If, moreover, $\rho_{i+1}<\mu_i$ $(i=1,\cdots,r-1)$ we say that $T_{\mu-\rho}$ is an echelon tableaux. Although not all parts of the Young theory hold for echelon tableaux, the author shows that much of it can be generalized. In particular, he associates a representation of the full linear group $GL(n)$ with the echelon tableaux $T_{\mu-\rho}$ and calculates its character. He also treats composite tableaux whose components are echelon tableaux.

R. M. Thrall (Ann Arbor, Mich.).

Wallace, Andrew H. A note on the Capelli operators associated with a symmetric matrix. Proc. Edinburgh Math. Soc. (2) **9**, 7–12 (1953).

The author gives new derivations for the modified Capelli theorem and Gårding's theorem for a symmetric matrix X. The derivation is based on a factorization $X=Y'Y$ and

corresponding relations between differential operators and polarizations in the elements x_{ij} of X and y_{ij} of Y.

R. M. Thrall (Ann Arbor, Mich.).

Inonu, E., and Wigner, E. P. On a particular type of convergence to a singular matrix. Proc. Nat. Acad. Sci. U. S. A. **40**, 119–121 (1954).

It is noted that a result on matrices used by the authors in same Proc. **39**, 510–524 (1953) [these Rev. **14**, 1061] is unsound but becomes valid in the presence of a certain additional assumption by virtue of the following theorem. If u and w are $n\times n$ matrices such that u has rank $r<n$ while $u+\epsilon w$ is non-singular except for isolated ϵ, then $u(u+\epsilon w)^{-1}u\rightarrow u$ as $\epsilon\rightarrow 0$ if and only if the coefficient of ϵ^{n-r} in the determinant $|u+\epsilon w|$ does not vanish.

I. E. Segal (New York, N. Y.).

Jackson, James R. On the existence problem of linear programming. Pacific J. Math. **4**, 29–36 (1954).

The existence problem of linear programming as understood by the author is as follows: Given a matrix A and a vector b, is there a vector x with nonnegative components such that $Ax=b$? The author is able to answer this question in the case that there is a row vector w such that wA has all positive elements. He notes that there is such a vector w in "a great many applications." He begins his analysis by converting the problem to a problem $Dx=e$, where the columns of the matrix D and the one column of e all sum to unity. He then writes $H_{ij}=D_{ij}-e_i$ and considers the two-person zero-sum game defined by H. Then, except for trivial exceptions, a necessary and sufficient condition that the linear programming problem have a solution is that the value of the game defined by H be zero.

J. M. Danskin (Washington, D. C.).

Carbonaro, Carmela Marletta. La funzione inversa $y=x^{-1}$ in un'algebra complessa semi-semplice. Boll. Accad. Gioenia Sci. Nat. Catania (4) **2**, 195–201 (1953).

Abstract Algebra

Benado, Mihail. Les ensembles partiellement ordonnés et le théorème de raffinement de O. Schreier. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 585–591 (1952). (Romanian. Russian and French summaries)

Let S be any partially ordered set satisfying the following condition for fixed $a\in S$ and all b, c, $d\in S$: if $a\leqq b\leqq d$ and $a\leqq c\leqq d$, then there exist a', $d'\in S$ with $a\leqq a'\leqq b\leqq d'\leqq d$ and $a'\leqq c\leqq d'$ such that the order-isomorphisms $d'/b\cong c/a'$ and $d'/c\cong b/a'$ hold. Theorem: In S, any two chains with common extremities $a\leqq d$ have isomorphic refinements. The essential part of the proof is stated to follow exactly that of Schreier [Abh. Math. Sem. Hamburg. Univ. **6**, 300–302 (1928)]. It is observed that various known results are special cases of this theorem.

P. M. Whitman.

Jakubík, Ján. Uniqueness of decomposition of a lattice in a direct product. Mat.-Fyz. Sborník Slovensk. Akad. Vied Umení **1**, 45–50 (1951). (Slovak. Russian summary)

Decomposition of a lattice into a direct product of irreducible lattices is unique, even if the lattice does not have O or I and the number of factors is infinite. [Cf. G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ. v. 25, rev. ed., New York, 1948, Problem 11; these Rev. **10**, 673. Related results are given by J. Hashimoto, Ann. of Math. (2) **54**, 315–318 (1951); these Rev. **13**, 201; and T. Nakayama, Math. Japonicae **1**, 49–50 (1948); these Rev. **10**, 279.] The key to the proof is as follows. Suppose $L=\prod_{\lambda\in\Lambda}L_\lambda$. If $w\in L$, $\alpha\in\Lambda$, and $S_\alpha\subset L_\alpha$, denote by $S_\alpha(w)$ the set of elements of the direct product which have α-component belonging to S_α and other components the same as those of w. If L has two decompositions $L=\prod_{\mu\in M}A_\mu$ and $L=\prod_{\nu\in N}B_\nu$ and if $\alpha\in M$, $\beta\in N$, denote by A^* the set of β-components of $A_\alpha(w)$. Then $A^*(w)=A_\alpha(w)\cap B_\beta(w)$.

P. M. Whitman.

Jordan, Pascual. Zur Theorie der nichtkommutativen Verbände. Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. **1953**, 59–64 (1953).

The author previously defined non-commutative ("skew") lattices [Arch. Math. **2**, 56–59 (1949); these Rev. **11**, 309]. He now gives several non-trivial examples of such. Each example has as elements chains, of specified length, of elements of an ordinary lattice, with unusual rules for the meet and join of chains. There is some discussion of the properties of $\leqq$ and $\geqq$, defined in skew-lattices as $a\leqq b$ means $b\cap a=a$ and $a\geqq b$ means $a\cup b=a$.

P. M. Whitman (Silver Spring, Md.).

Jordan, Pascual, und Witt, Ernst. Zur Theorie der Schrägverbände. Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. **1953**, 223–232 (1953).

[Cf. the preceding review.] Further examples of skew-lattices, based on pairs of isotone mappings of a lattice or of a set mapped into a lattice. It is observed that there are four different obvious definitions of inclusion of a in b in a skew-lattice: $b\cap a=a$ ("anterior strong"); $b\cup a=b$ ("posterior strong"); $a\cup b=b$ ("anterior weak"); $a\cap b=a$ ("posterior weak"). Still others can be defined, of the type: a included in b if $b\cap x=a$ has a solution. The relationships of the different types of inclusion are examined. Some open questions are posed.

P. M. Whitman.

Schuff, Hans Konrad. Zur Darstellung von Polynomen über Verbänden. Math. Nachr. **11**, 1–4 (1954).

Given a set $\{x_\iota\}$ of indeterminates, and a lattice L, the author constructs, with canonical forms, the lattice of lattice-polynomials in the x_ι over L. [The rules bear a marked resemblance to those found by the reviewer for elements of free lattices [cf. Ann. of Math. (2) **42**, 325–330 (1941); **43**, 104–115 (1942); these Rev. **2**, 244; **3**, 261] and generalized by Sorkin [Mat. Sbornik N.S. **30**(72), 677–694 (1952); these Rev. **13**, 901].]

P. M. Whitman.

Vorob'ev, N. N. On congruences of algebras. Doklady Akad. Nauk SSSR (N.S.) **93**, 607–608 (1953). (Russian)

An equivalence relation can be interpreted as a set of ordered pairs of elements. Hence, a congruence relation ϕ on an algebra A can be interpreted in terms of subalgebras of the direct product $A\times A$. But the set of all ϕ is not a sublattice of the lattice of all subalgebras of $A\times A$.

P. M. Whitman (Silver Spring, Md.).

Kleinfeld, Erwin. Right alternative rings. Proc. Amer. Math. Soc. **4**, 939–944 (1953).

A right alternative ring R is a system with an addition and a multiplication in which the addition is an abelian group, both distributive laws hold, and also the right alternative law $(xy)y=x(yy)$. We define the associator (x, y, z) by

the rule $(x, y, z) = (xy)z - x(yz)$. It is shown here that R also satisfies the left alternative law $(xx)y = x(xy)$ and so is an alternative ring if two conditions are satisfied: (1) $2z=0$ in R implies $z=0$; (2) if x, y, z are any of certain expressions in two parameters u and v, then $(x, y, z)^2=0$ implies $(x, y, z)=0$. A similar result was found by Skornyakov [Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, 177–184 (1951); these Rev. 12, 669] where the second condition was replaced by the stronger assumption that R contains no divisors of zero. *Marshall Hall, Jr.* (Columbus, Ohio.).

Northcott, D. G., and Rees, D. Reductions of ideals in local rings. Proc. Cambridge Philos. Soc. 50, 145–158 (1954).

Dans un anneau commutatif A un idéal $\mathfrak{b}$ est appelé une réduction d'un idéal $\mathfrak{a}$ si $\mathfrak{b}\subset\mathfrak{a}$ et s'il existe un entier n tel que $\mathfrak{b}\mathfrak{a}^n=\mathfrak{a}^{n+1}$. Les idéaux $\mathfrak{a}$ et $\mathfrak{b}$ ont alors mêmes propriétés "asymptotiques", en particulier même dimension et même multiplicité. Dans le cas d'un anneau local A, les auteurs démontrent l'existence de réductions minimales de $\mathfrak{a}$; des éléments de $\mathfrak{a}$ en nombre convenable engendrent une réduction minimale de $\mathfrak{a}$ s'ils sont "suffisamment généraux". Un idéal n'admettant d'autre réduction que lui même est appelé basique; diverses caractérisations des idéaux basiques sont données, et font intervenir la notion d'éléments analytiquement indépendants d'un anneau local. La somme de tous les idéaux d'un anneau local A qui admettent un idéal donné $\mathfrak{a}$ pour réduction est un idéal $\bar{\mathfrak{a}}$ qui admet $\mathfrak{a}$ pour réduction; c'est aussi, lorsque $\mathfrak{a}$ ne se compose pas uniquement de diviseurs de zero, l'ensemble des éléments de A qui dépendent analytiquement [ou "intégralement"; cf. Prüfer, J. Reine Angew. Math. 168, 1–36 (1932)] de $\mathfrak{a}$. Enfin les auteurs généralisent à une classe plus vaste d'idéaux la formule d'associativité des multiplicités dans les anneaux à noyau [cf. Chevalley, Trans. Amer. Math. Soc. 57, 1–85 (1945); ces Rev. 7, 26]. *P. Samuel.*

Higgins, P. J. Lie rings satisfying the Engel condition. Proc. Cambridge Philos. Soc. 50, 8–15 (1954).

The author is concerned with a generalization of Engel's theorem to Lie rings. He inquires whether a Lie ring $\mathfrak{L}$ must be nilpotent if ad x is nilpotent for each x in $\mathfrak{L}$ and the indices of nilpotency are bounded. He then proves that with certain assumptions on the additive orders of elements of $\mathfrak{L}$ the hypothesis above implies the equivalence of solvability and nilpotency and also the nilpotency of $\mathfrak{L}$ when the indices of nilpotency are bounded by four. The method is essentially the use of polynomial identities in Lie rings. In particular, no attempt is made to determine the structure of the enveloping associative ring of the inner derivations of $\mathfrak{L}$, the method by which Jacobson [C. R. Acad. Sci. Paris 234, 579–581 (1952); these Rev. 13, 618] obtained a stronger result under the additional assumption of a chain condition. *W. G. Lister* (Providence, R. I.).

Širšov, A. I. On the representation of Lie rings as associative rings. Uspehi Matem. Nauk (N.S.) 8, no. 5(57), 173–175 (1953). (Russian)

Kuročkin [Mat. Sbornik N.S. 28(70), 467–472 (1951); these Rev. 12, 799] claimed that any Lie ring with operators Ω admits a faithful Ω-representation in an associative ring. Later [ibid. 30(72), 463 (1952); these Rev. 13, 719] he acknowledged an error in the proof. In the present paper the author gives a counter-example. For the Lie ring he takes a certain 13-dimensional algebra over the field of two elements with all products of three elements equal to 0; Ω is taken to be a three-dimensional trivial algebra with a unit element adjoined. It is noted at the end that the theorem is true if Ω is a principal ideal ring, the paper of Lazard [C. R. Acad. Sci. Paris 234, 788–791 (1952); these Rev. 13, 719] being cited. *I. Kaplansky* (Chicago, Ill.).

Širšov, A. I. Subalgebras of free Lie algebras. Mat. Sbornik N.S. 33(75), 441–452 (1953). (Russian)

It is shown that every subalgebra A of a free Lie algebra L over a field F is again a free Lie algebra. He uses the basis B for free Lie rings given by the reviewer [Proc. Amer. Math. Soc. 1, 575–581 (1950); these Rev. 12, 388]. The proof depends upon constructing a set M of free Lie generators for A. Take a set T which forms a basis for A regarded as a vector space over F. We can modify T so that no two elements of T have as their element of highest order the same element of B. Then order T on the elements of highest order and take M as the elements of T not generated by predecessors. He then proves that the elements of M are free Lie generators for A. *Marshall Hall, Jr.*

Karpelevič, F. I. Classification of the simple subalgebras of the real forms of classical algebras. Doklady Akad. Nauk SSSR (N.S.) 93, 613–616 (1953). (Russian)

In a previous paper [same Doklady 85, 1205–1208 (1952); these Rev. 14, 245] the author announced the classification of the simple subalgebras of the real forms of the Lie algebra of complex matrices of trace zero. This work is now extended to cover the other two families of classical algebras: the skew-symmetric and skew-symplectic matrices. Recapitulation of the theorems would be of the same order of magnitude as the paper itself and is perhaps best deferred until the publication of the detailed results.

I. Kaplansky (Chicago, Ill.).

Osima, Masaru. Supplementary remarks on the Schur relations for a Frobenius algebra. J. Math. Soc. Japan 5, 24–28 (1953).

The paper is a continuation of an earlier investigation of the author [same J. 4, 1–13 (1952); these Rev. 14, 349]. Some further relations for the coefficients of the indecomposable components of the regular representations of a Frobenius algebra are given. *R. Brauer.*

Kaplansky, Irving. Infinite-dimensional quadratic forms admitting composition. Proc. Amer. Math. Soc. 4, 956–960 (1953).

Let A be a vector space over a field F. A function $g(x)$ from A to F is a quadratic form if $g(\lambda x)=\lambda^2 g(x)$ for $\lambda \in F$, $x \in A$ and if also $f(x, y)=g(x+y)-g(x)-g(y)$ is bilinear in x and y. It is shown here without any assumption on the dimension of A, that if $g(x)g(y)=g(xy)$ where xy is a bilinear product from $A\times A$ to A, then the theorem of Hurwitz holds and A is of dimension 1, 2, 4, or 8 over F and that the composition comes from either F, a quadratic, a quaternion, or Cayley-Dickson algebra over F. The only exception to this occurs when $f(x, y)$ is identically zero and then A is a purely inseparable quadratic extension of F, being of characteristic two and here $g(x)=x^2$.

In its essentials the proof consists of taking the bilinear product xy to make A into a ring over F. With this product or an isotopic product A can in general be taken to have a unit. With this product A is shown to be alternative and its elements quadratic over F. Next A is shown to be a simple ring. Quadratic associative rings must be of dimension 1, 2, or 4 over F and a recent result of Kleinfeld

[Ann. of Math. (2) **58**, 544–547 (1953); these Rev. **15**, 392] shows that a simple alternative ring, not associative, is a Cayley-Dickson algebra of dimension 8 over F.

Marshall Hall, Jr. (Columbus, Ohio).

Krull, Wolfgang. Zur Galoisschen Theorie der arithmetischen Körper. Math. Ann. **126**, 239–252 (1953).

This paper may be regarded as a supplement to an earlier work (to be denoted by K1) of the author [Math. Ann. **121**, 446–466 (1950); these Rev. **12**, 796]. We shall use without further explanation the notations of the review of the earlier paper.

An ideal in R or R_s is called permanent if it is the contraction of an ideal in S. It is shown that the mapping $q_s \to q_s \cap R$ is 1-1 from the set of all permanent primary ideals belonging to p_s to the set of all permanent primary ideals belonging to p. It had been proved in K1 that for such q_s, $R_s/q_s \cong R/(q_s \cap R)$, in particular, $R_s/p_s \cong R/p$. (It has subsequently been proved by Nagata [Nagoya Math. J. **5**, 45–57 (1953); these Rev. **14**, 529] that $R_s P$ has P_s as a primary component.)

If q_s is now any primary ideal belonging to p_s, then $R_t q_s \cap R_s = q_s$, and $R_t/R_t q_s$ is a free module over R_s/q_s on m generators, where $m = [K_t : K_s]$. If R_s is Noetherian, then a necessary and sufficient condition that the ideals $R_t q_s$ include all primary ideals belonging to p_t is that the quotient ring of R_s with respect to p_s be a discrete valuation ring.

In K1, higher ramification groups were not defined. In the present paper a decreasing sequence $\{G_{vi}\}$ of normal subgroups of G_v is defined as follows: G_{vi} consists of those $\sigma \in G_v$ such that $\sigma a - a \in \bar{p}^{i+1}$ for all $a \in \bar{p}$. For the case of a discrete valuation ring this is the classical definition. Moreover, as in the classical case, $G_{vi}/G_{v,i+1}$ $(i \geqq 1)$ is Abelian of type $(p, p, \cdots, p)$, where p is the characteristic of $\bar{N}$, and if $\bigcap_{k=1}^{\infty} \bar{p}^k = (0)$, then G_{vi} ultimately vanishes.

There is some discussion of alternative definitions of the higher ramification groups, and questions meriting further investigation are indicated.

I. S. Cohen.

Kasch, Friedrich. Über den Endomorphismenring eines Vektorraumes und den Satz von der Normalbasis. Math. Ann. **126**, 447–463 (1953).

The normal-basis theorem which asserts that a normal extension K of finite degree of a field H always possesses a basis of the form $\{vg\}$ where $v \in K$ and g varies over the Galois group G of K over H can be viewed in other ways. One of these is as follows. Consider K as a K module; then the elements of G induce endomorphisms of this module; likewise do the left multiplications and right multiplications by elements of K; denote the latter two by K^l and K^r and let H^l and H^r denote the corresponding endomorphisms obtained by using elements of H. Let $R = (H^r, G)$ be the endomorphism ring of K generated by H^r and G. Then the normal basis theorem can be stated as: R and K are operator isomorphic when considered as R-modules.

Let K be a simple ring with minimum condition on right ideals, H a Galois subring (which demands that it satisfy certain three properties) over which K is finite-dimensional and let G be the Galois group of K over H. Let Z be the center of K and T the centralizer of H in K. Amongst similar results the author proves the following typical theorems. (1) If K over H has only outer automorphisms, i.e. if $Z = T$ or $T \subset H$, then $[K:H]_r = [R:H^r]_r$ and R is operator isomorphic to K. (2) If $Z = T$ or $T \subset H$, then there are $n = [K:H]_r$ automorphisms $g_1, \cdots, g_n$ of K over H and an element $k \in K$ so that the elements $kg_1, \cdots, kg_n$ form a right basis of K over H. (3) Every Galois division algebra extension K of H possesses two generating elements over H. If $Z = T$ or $T \subset H$ then there exists one generating element of K over H.

I. N. Herstein (Philadelphia, Pa.).

Nagata, Masayoshi, Nakayama, Tadasi, and Tuzuku, Tosiro. On an existence lemma in valuation theory. Nagoya Math. J. **6**, 59–61 (1953).

Let K be a field which either is transcendental over its prime field or else is of characteristic 0; let L be a separable extension of K other than K itself. The authors give a simple proof of the fact that there exist infinitely many valuations of L which are of degree 1 over K.

E. Kolchin (New York, N. Y.).

Cartan, Henri. Extension du théorème des "chaînes de syzygies". Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11**, 156–166 (1952).

J. L. Koszul [Colloque de topologie (espaces fibrés), Bruxelles, 1950, Thone, Liège, 1951, pp. 73–81; these Rev. **13**, 109] has shown that the source of Hilbert's theorem on the finiteness of the chains of syzygies for a homogeneous polynomial ideal [Math. Ann. **36**, 473–534 (1890)] is essentially the following homology property of the polynomial ring $A = K[x_1, \cdots, x_n]$ in n variables over the field K: there exists a free acyclic (in positive dimensions) A-complex G whose 0-dimensional homology group $H_0(G)$ coincides with K and whose m-dimensional chain groups G_m vanish for $m > n$. In the terminology of the forthcoming book on homological algebra by the author and S. Eilenberg, this implies the vanishing of the groups $\mathrm{Tor}_m(M, K)$ for all unitary right A-modules M and all $m > n$. In fact, these groups are the homology groups $H_m(M \otimes G)$ of the tensor product (relative to A) complex $M \otimes G$. The connection with Hilbert's theorem is due to the basic fact that $\mathrm{Tor}_m(M, K)$ can also be computed from any free acyclic right A-complex F with $H_0(F) = M$; namely as the homology group $H_m(F \otimes K)$. In the situation of Hilbert's theorem, the conclusion $\mathrm{Tor}_m(M, K) = (0)$ for $m > n$ can be strengthened to (γ) of the theorem quoted below. Hilbert's construction of his "abgeleitete Gleichungssysteme" for a homogeneous ideal M is essentially the construction of a complex F as above, which starts with an exact sequence such as in (γ). Hence the theorem below, together with the final result securing the validity of condition (α'), constitutes a generalization of Hilbert's theorem, with Koszul's result occupying an intermediate place.

Let A be a ring (not necessarily commutative) with an identity element, and let I be a two sided ideal of A such that $K = A/I$ is a division ring, while the intersection of all powers of I is (0). Furthermore, make one of the following two sets of assumptions. Case 1: A is a graded ring, and I is the ideal of all elements whose components of degree 0 are equal to 0. In this case, assume that M is a unitary graded right A-module, so that $M_p \cdot A_q \subset M_{p+q}$. Case 2: A satisfies the ascending chain condition for right ideals, and I is contained in the Jacobson radical of A. In this case, assume that M is a unitary finitely generated right A-module. Theorem: In case 1 or case 2, and with n any positive integer, the following conditions are equivalent: (α) $\mathrm{Tor}_{n+1}(M, K) = (0)$; (α') $\mathrm{Tor}_m(M, K) = (0)$ for all $m > n$; (β) there exists a free acyclic right A-module F such that $H_0(F) = M$ and $F_m = (0)$ for $m > n$; (γ) for every exact sequence $M_{n-1} \to M_{n-2} \to \cdots \to M_1 \to M_0 \to M \to (0)$, where the M_i are free right A-modules, the kernel of the first homomorphism from the left is A-free. (In case 1, the F_i and

M_i of (β) and (γ) are graded, while in case 2 they are of finite rank.)

Finally, it is shown (essentially by following Koszul's construction of a free acyclic complex G as described in the beginning) that (α') holds whenever A and I satisfy the following conditions: there is a sequence of subrings of A, $A_0 \subset A_1 \subset \cdots \subset A_n = A$, such that the canonical homomorphism $A \to A/I$ maps A_0 isomorphically onto A/I, and there are elements $x_1, \cdots, x_n$ in the center of A such that $x_m \in A_m$ and every element of A_m can be written uniquely in the form $ux_m + v$, with $u \in A_m$ and $v \in A_{m-1}$. Clearly, these conditions are adapted to the three typical cases of polynomial rings, formal power series rings, and rings of convergent (in some neighborhood of 0) power series over a complete valuated field. Furthermore, each of these cases falls under one of the headings (case 1 or case 2) above.

G. Hochschild (Urbana, Ill.).

Fujiwara, Izuru. On the Duffin-Kemmer algebra. Progress Theoret. Physics 10, 589–616 (1953).

The main novelty of this treatment is that it is purely algebraic, being based solely on the relations

$$\beta_\mu \beta_\lambda \beta_\nu + \beta_\nu \beta_\lambda \beta_\mu = g_{\mu\lambda} \beta_\nu + g_{\nu\lambda} \beta_\mu$$

(the algebra in question being the enveloping algebra of the $\beta_1, \beta_2, \cdots, \beta_n$), and avoids the Γ-formalism of Harish-Chandra [Proc. Roy. Soc. London. Ser. A. 186, 502–525 (1946); these Rev. 8, 302]. An explicit construction is given for bases for the simple constituents of the algebra. These lead to tensor representations of the group of real orthogonal transformations on the n-dimensional β-space. Applications are made to the correspondence between the particle and wave forms of meson theory and to developments due originally to Klein [Physical Rev. (2) 82, 639–646 (1951); these Rev. 13, 412]. *I. E. Segal* (New York, N. Y.).

Theory of Groups

Pták, Vlastimil. Immersibility of semigroups. Čehoslovack. Mat. Ž. 2(77), 247–271 (1952). (Russian. English summary)

"Extract from a paper published by the author" [Acta Fac. Nat. Univ. Carol., Prague no. 192 (1949); these Rev. 12, 155].

Teissier, Marianne. Sur les demi-groupes ne contenant pas d'élément idempotent. C. R. Acad. Sci. Paris 237, 1375–1377 (1953).

(1) Let D be a demi-group admitting division on the left, and not containing an idempotent element. There exists a homomorphic image D' of D, containing an idempotent, if and only if D contains a sub-demi-group E which is unitary on the left ($u \in E$ and $xu \in E$ imply $x \in E$) and is such that $EaE \subseteq Ea$ for every $a \in D$. (2) An example is given of a demi-group containing two non-isomorphic minimal left ideals. *A. H. Clifford* (Baltimore, Md.).

Campaigne, Howard H. A lower limit on the number of hypergroups of a given order. J. Washington Acad. Sci. 44, 5–7 (1954).

If n is finite and greater than 3, the number of hypergroups of order n is at least $8 \cdot 11^{n-2}$. This is proved by induction, starting from the 8 hypergroups of order 2, adjoining one element at a time in eleven different ways, and showing that the eleven extensions are non-isomorphic. The number of all possible algebras with one binary operation under which products are non-null subsets, viz. $(2^n-1)n^2$, gives a crude upper bound to the number of hypergroups of order n. *H. A. Thurston* (Bristol).

Scott, W. R. The number of subgroups of given index in nondenumerable Abelian groups. Proc. Amer. Math. Soc. 5, 19–22 (1954).

A group of order A, where $A > \aleph_0$, and $A \geqq B \geqq \aleph_0$ has exactly 2^A subgroups of index B and order A; and their intersection is $\{0\}$. Further, there is a set of 2^A subgroups H of index B and order A such that the groups G/H are all isomorphic; in fact each is the direct sum of groups which are either all cyclic of prime order or all p^∞ groups (p not necessarily fixed). This generalizes a previous result of the same author [Amer. J. Math. 74, 187–197 (1952), Theorem 9; these Rev. 13, 721]. *H. A. Thurston* (Bristol).

Haimo, Franklin. Some non-abelian extensions of completely divisible groups. Proc. Amer. Math. Soc. 5, 25–28 (1954).

Let G be a group, and H a subgroup of G such that $H^n = H$ for all positive integers n. Suppose also that H lies between the successive terms Z_d and Z_{d+1} of the upper central series of G. Then if K is a subgroup of G intersecting H in Z_d, and maximal with respect to this property, the normaliser of K in G is the product HK. *Graham Higman.*

Baer, Reinhold. Das Hyperzentrum einer Gruppe. III. Math. Z. 59, 299–338 (1953).

In the terminology of the first two papers of the series [Acta Math. 89, 165–208 (1953); Arch. Math. 4, 86–96 (1953); these Rev. 15, 395, 396] the problem is the characterisation of finitely generated upper hypercentral subgroups. First, two theorems are proved concerning finitely generated upper nilpotent groups. One says that in such a group there is a set of characteristic subgroups whose indices are prime powers p^x with bounded x, and which intersect in 1; the other that for finitely generated soluble groups the author's concepts of nilpotence and upper nilpotence coincide. Turning to his main theme, the author defines $\Phi^*(G)$, for any group G, to be the intersection of its maximal subgroups of finite index. Then a finitely generated hypercentral subgroup is upper hypercentral if and only if some iterate of Φ^* maps it on 1. Here the hypothesis of hypercentrality can be weakened, provided that the others are suitably strengthened. For instance, call the subgroup N of G weakly hypercentral if N/M is a hypercentral subgroup of G/M whenever M is a subgroup of finite index in N which is normal in G. Then a weakly hypercentral subgroup is upper hypercentral provided that it is soluble, and that every normal subgroup of G contained in it is finitely generated; and there are several similar alternative criteria.

Graham Higman (Manchester).

Borevič, Z. I. On an Abelian group with operators. Doklady Akad. Nauk SSSR (N.S.) 91, 193–195 (1953). (Russian)

Let $G = F/R$ be a group which is presented as a factor group of a free group F on generators x_i. Define four G-modules Z, O, W, and R_0 as follows: Z is the additive group of integers, G operating simply; O is the group ring of G with integer coefficients; $W = \sum_i Odx_i$ is the free G-module with a base consisting of symbols dx_i, one for each generator x_i; and $R_0 = R/[R, R]$ is the factor commu-

tator group of the relation subgroup R, with G operating by conjugation. Introducing the notation $r \to \bar{r}$ and $u \to u^*$ for the natural maps $R \to R_0$ and $F \to G$ we define a sequence of G-homomorphisms

$$(1) \qquad 0 \to R_0 \xrightarrow{g} W \xrightarrow{f} O \xrightarrow{e} Z \to 0$$

by putting $e(1)=1$; $f(dx_i)=x_i^*-1$; and

$$g(\bar{r})=\sum_i (\partial r/\partial x_i)^* dx_i.$$

Here the Fox partial derivatives $\partial r/\partial x_i$ are the elements of the group ring of F defined by the identity

$$r-1=\sum_i (\partial r/\partial x_i)(x_i-1).$$

The author's main result is that the sequence (1) is exact. (Actually, he defines W differently, as the splitting module of a 2-cocycle of the extension $G \simeq (F/[R, R])/R_0$, so that the sequence is obviously exact, and then must prove W is G-free.) He then mentions several homology and cohomology isomorphisms of the cup product reduction type which involve a dimension shift of two, and which are immediate consequences of the existence of an exact sequence of type (1) which links R_0 to Z by way of two G-free modules. *J. T. Tate* (New York, N. Y.).

Faddeev, D. K. On a theorem of the theory of homologies in groups. Doklady Akad. Nauk SSSR (N.S.) **92**, 703–705 (1953). (Russian)

If $\mathfrak{H}$ is a subgroup of finite index v of a group $\mathfrak{G}$, if $\mathfrak{a}$ is a $\mathfrak{G}$-module in which division by v is always possible and unique, then the cohomology module $H^n(\mathfrak{G}, \mathfrak{a})$ is isomorphic with a direct summand of $H^n(\mathfrak{H}, \mathfrak{a})$. *E. R. Kolchin.*

Berman, S. D. On a necessary condition for isomorphism of integral group rings. Dopovidi Akad. Nauk Ukrain. RSR **1953**, 313–316 (1953). (Ukrainian. Russian summary)

Let G be a group of finite order, let K be a number field. Letting n be the smallest integer >0 such that $x^n=1$ for every x in G, and letting ϵ be a primitive nth root of unity, the author calls elements x, y of G K-conjugate if x is conjugate to a power y^m such that $\varphi(\epsilon)=\epsilon^m$ for some automorphism φ of $K(\epsilon)$ over K. Theorem. Let G, H be groups of finite order such that their group rings (over the ring of rational integers) are isomorphic. Then, for every number field K, there is a one-to-one correspondence between the set of K-conjugate classes of G and that of H, such that corresponding classes have the same number of elements. *E. R. Kolchin* (New York, N. Y.).

Zavalo, S. T. Γ-free operator groups. Mat. Sbornik N.S. **33**(75), 399–432 (1953). (Russian)

Let Σ be a semigroup with elements $\alpha, \beta, \gamma, \cdots$ including an identity ϵ. Then elements x_i of a set M and $x_i\epsilon=x_i$, $x_i\alpha$, $\alpha\in\Sigma$, may be taken as generators of a free group F. Then F will admit Σ as a set of operators if for u any element of M or the inverses of M we set $u\epsilon=u$, $(u\alpha)\beta=u(\alpha\beta)$, $u^{-1}\alpha=(u\alpha)^{-1}$. We will have the same relations for any element of F by defining $(u_1u_2\cdots u_t)\alpha=u_1\alpha u_2\alpha\cdots u_t\alpha$. Such a group is called a Σ-free group. It is shown that every group admitting Σ as operators is a homomorphic image of a Σ-free group.

More attention is given to the case in which Σ is a group Γ. Here for g some fixed element of F the set of elements $g^{-1}(g\alpha)$, $\alpha\in\Delta$, a subgroup of Γ, will generate a subgroup $A(g, \Delta)$. If Δ contains elements of finite order, it is easily seen that $A(g, \Delta)$ is not operator-free. As a general theorem it is shown that any admissible subgroup U of the Γ-free group F is the free product of a Γ-free subgroup H and subgroups $A(g, \Delta)$. *Marshall Hall, Jr.*

Taylor, Robert L. Compound group extensions. I. Continuations of normal homomorphisms. Trans. Amer. Math. Soc. **75**, 106–135 (1953).

This question, in the theory of non-commutative groups, arises from topology. For a "normal" homomorphism ϕ: K into G, that is, with image normal in G, a "continuation" is a pair $(E, \bar{\phi})$ where $E \supset K$ and $\bar{\phi}$: E onto G is an extension of ϕ with the same kernel, X. A continuation induces a homomorphism θ: G into A, the group of automorphisms of A carrying X into itself, modulo those inner automorphisms induced by X. Generally, normal ϕ: K into G, and θ: G into A constitute a "pseudo-module" (ϕ, θ) provided $\theta\phi$ is the natural map K into A. A continuation $(E, \bar{\phi})$ that induces θ, provided such exists, is an "extension" of the ps-module (ϕ, θ).

A crossed-module [J. H. C. Whitehead, Bull. Amer. Math. Soc. **55**, 453–496 (1949); these Rev. **11**, 48] is a ps-module for which the kernel X of ϕ lies in the center Z of K. A Q-kernel [Eilenberg and MacLane, Ann. of Math. (2) **48**, 326–341 (1947); these Rev. **9**, 7] is a ps-module for which ϕ is trivial. The author sets forth carefully the correspondence between $H^3(Q, X)$ and the (isomorphism classes of) extensions of cr-modules and Q-kernels, and the theory of the obstruction. These results are extended to ps-modules by the following device. Let $\bar{A}$ be the group of automorphisms of K that carry X into itself, with the natural maps c: K into $\bar{A}$, λ: $\bar{A}$ into A. The graph [cf. Baer, Math. Z. **38**, 375–416 (1934)] $\Gamma \subset G \oplus A$ is the group of all (g, α) with $\theta g=\lambda\alpha$. The exact sequence 0, X, K, G, Q, 0 leads to an exact sequence 0, $X \cap Z$, K, Γ, Q, 0. For ϕ' the induced map K into Γ, and $\theta'(g, \alpha)=\alpha$, this yields a cr-module (ϕ', θ') whose extension theory is equivalent to that of the original ps-module (ϕ, θ). In fact, it is shown that the concept of ps-module is equivalent to that of a pair formed by a cr-module together with a suitably related subgroup of K. An additional result states that (ϕ, θ) admits a splitting extension $(E, \bar{\phi})$ just in case G splits over the image of ϕ. *R. C. Lyndon* (Ann Arbor, Mich.).

Taylor, Robert L. Compound group extensions. II. Trans. Amer. Math. Soc. **75**, 304–310 (1953).

Let $\{\rho_r: R_r \to R_{r+1}\}$ and $\{\sigma_r: S_r \to S_{r+1}\}$ be two (infinite) exact sequences, and $\{\beta_r: R_r \to S_r\}$ a commuting sequence of homomorphisms; throughout, the additional hypothesis is made that $\sigma_1\beta_1R_1=\sigma_1S_1$. Call this figure F, and F' the figure obtained by deleting $R_2\rho_1$, ρ_2, and β_2. This paper characterizes the set of (isomorphism classes of) "completions" F of a given figure F'. Replacing R_1 by a factor group and R_3 by a subgroup reduces the problem to the case where R_0, S_0, R_4 and S_4 are all trivial, as we may assume henceforth. If β_1 is an isomorphism of R_1 onto S_1, there is a unique solution with R_2 the appropriate graph $\Gamma \subset R_3 \oplus S_2$. With reference to this case, for S_1 in place of R_1, the general problem is now shown to reduce to that for continuations (in the sense of part I, reviewed above) of the naturally induced normal homomorphism $R_1 \to S_1 \to \Gamma$. Finally, if θ is given constituting, together with β_1, a pseudo-module, there corresponds naturally a pseudo-module (β_1', θ') with β_1': $R_1 \to \Gamma$, and the completions of F' correspond one-to-one with the extensions of (β_1', θ'). *R. C. Lyndon.*

Nagai, Osamu. Supplement to "Note on Brauer's theorem of simple groups". Osaka Math. J. 5, 227–232 (1953).

The author extends the result of his paper in the same J. 4, 113–120 (1952) [these Rev. 14, 843] to the case $n=p+2$, t odd. *R. Brauer* (Cambridge, Mass.).

Shephard, G. C., and Todd, J. A. Finite unitary reflection groups. Canadian J. Math. 6, 274–304 (1954).

In two earlier papers [Proc. London Math. Soc. (3) 2, 82–97 (1952); Canadian J. Math. 5, 364–383 (1953); these Rev. 13, 968; 14, 1060] the first author described the regular complex polytopes and their symmetry groups, which are generated by unitary reflections. (A unitary reflection is a unitary transformation of finite period that leaves invariant every point of a certain hyperplane.) In the present paper the authors make use of the existing literature on collineation groups in order to complete the enumeration of finite groups generated by unitary reflections. As might be expected by analogy with the case of real reflections, there are some groups not related to regular polytopes but only to semi-regular polytopes. For instance, $G(m, p, n)$, of order $m^n n!/p$, is the symmetry group of the complex polytope whose vertices are

$$(\theta^{\nu_1}, \theta^{\nu_2}, \cdots, \theta^{\nu_n}),$$

where θ is a primitive mth root of unity and $\sum \nu_i \equiv 0 \pmod{p}$. Apart from this family and the real groups [Coxeter, Ann. of Math. (2) 35, 588–621 (1934), p. 618], the n-dimensional reflection groups are found to be as follows: nineteen for $n=2$, four for $n=3$, three for $n=4$, one for $n=5$, and one for $n=6$. A set of generating relations is obtained for each group.

In the second part of the paper (beginning on page 282) the authors prove that a finite group of unitary transformations on n variables is generated by reflections if and only if it possesses a set of n algebraically independent invariant forms such that every invariant form of the group is expressible as a polynomial in the forms of the set. The degrees, m_i+1, of the basic invariant forms, are found to have many of the same properties as in the real case [Coxeter, Duke Math. J. 18, 765–782 (1951); these Rev. 13, 528]. In particular, the order of the group is $\prod(m_i+1)$. *H. S. M. Coxeter* (Toronto, Ont.).

Papy, Georges. Groupes différentiels gradués et différentielle extérieure. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 39, 293–308 (1953).

If for each integer ν, G^ν is an additive group (with operators) and φ^ν is a homomorphism of G^ν into $G^{\nu+1}$, the family (G^ν, φ^ν) is a "chain"; thus the concept of chain is equivalent to the concept of graded group with homogeneous endomorphism of degree 1. A chain is "differential" (is "exact") if $\varphi^\nu G^\nu$ is a subgroup of (is equal to) the kernel of $\varphi^{\nu+1}$. The author derives some simple results about chains; for example, every differential chain such that, for each ν, every subgroup of G^ν is a direct summand, is a homomorphic image of an exact chain. As an application he deduces Poincaré's theorem (local version) that every closed differential form is exact. *E. R. Kolchin* (New York, N. Y.).

Osima, Masaru. On the induced characters of groups of finite order. Math. J. Okayama Univ. 3, 47–64 (1953).

An outline of this paper appeared in the Proc. Japan Acad. 28, 243–248 (1952); these Rev. 14, 351. The present paper gives the details of the proofs. *R. Brauer.*

Berezin, F. A. Linear finite-dimensional representations of Lie groups with a commutative radical. Doklady Akad. Nauk SSSR (N.S.) 93, 759–761 (1953). (Russian)

The first section gives a construction of a Lie group with a commutative radical (c.r.). Let K be a semi-simple group and Γ its representation on a space $\mathfrak{N}$. The elements of the group G are pairs $(k, \mathfrak{n})$, $k \in K$, $\mathfrak{n} \in \mathfrak{N}$ with the law $(k_1, \mathfrak{n}_1)\cdot(k_2, \mathfrak{n}_2) = (k_1 k_2, \Gamma(k_2^{-1})\mathfrak{n}_1+\mathfrak{n}_2)$. By means of this construction one may obtain a group which is locally isomorphic to an arbitrary group with c.r. and isomorphic to a group which is not a direct product of a group of the same type and a commutative group. The second section is devoted to the representation of a group with c.r. If ϕ is a representation of the group whose vector functions (v.f.) over $\mathfrak{N}$ are polynomials of degree $\leqq m$, m is called the degree of the v.f. The main theorem states that if ϕ is an arbitrary representation of a group with c.r., there exist representations $\phi_1, \cdots, \phi_n$ and invariant subspaces $R' \subset R$ of the space $R\phi_1 \dotplus \phi_2 \dotplus \cdots \dotplus \phi_n$ such that the representation generated by $\phi_1 \dotplus \phi_2 \dotplus \cdots \dotplus \phi_n$ in the factor space R/R' is equivalent to ϕ. Another theorem deals with a canonical representation of the algebra $\mathfrak{G}$ of the group G. If a representation has the form

$$\begin{bmatrix} \varphi_1 & \mathfrak{B}_1 & 0 & \cdots & \cdot & 0 \\ 0 & \varphi_2 & \mathfrak{B}_2 & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \varphi_{s-1} & \mathfrak{B}_{s-1} \\ 0 & \cdot & \cdot & \cdots & 0 & \varphi_s \end{bmatrix}$$

in order that it should be canonical, it is necessary and sufficient that

$$\begin{bmatrix} \varphi_i & \mathfrak{B}_i & 0 \\ 0 & \varphi_{i+1} & \mathfrak{B}_{i+1} \\ 0 & 0 & \varphi_{i+2} \end{bmatrix}$$

be a canonical representation of $\mathfrak{G}$ of degree 2 for every i. Thus the problem of canonical representations of groups with c.r. is reduced to representations of degrees one and two. *M. S. Knebelman* (Pullman, Wash.).

Kanitani, Jōyō. Sur la fonction définissant le loi de composition des transformations d'un groupe de Lie. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. 28, 41–60 (1953).

A formal series expansion is given of the normal coordinates c^i of the product $c=ab$ of two elements a, b in terms of their normal coordinates a^i, b^i (everything local, of course). The set-up of the computations seems sound enough, but the numerous printing errors may discourage one from following the naturally very lengthy calculations. Both the final result and various intermediate identities may prove to be useful in related problems. *A. Nijenhuis* (Princeton, N. J.).

Have

★Koszul, J. L. Sur certains groupes de transformations de Lie. Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 137–141. Centre National de la Recherche Scientifique, Paris, 1953.

Let G be a Lie group and M a G-manifold, that is, a connected differentiable manifold on which G acts differentiably to the left. A basic form of M is one which is invariant under the action of G and is annihilated by the interior products $i(X)$, where X is an element of the Lie algebra $\mathfrak{g}$ of G. The basic forms of M constitute a subalgebra of the algebra of differential forms, which is stable under exterior differentiation. The space M_G of orbits is in general not a manifold. The author studies the question: Under what

conditions do the basic forms of M give the real cohomology of M_G under exterior differentiation.

A useful notion is that of a transversally fibered neighborhood of an orbit. A G-manifold is called regular, if every point has a compact isotropy group and if every orbit admits a transversally fibered neighborhood. The following theorems are proved: (1) If G is compact, every orbit of a G-manifold admits a transversally fibered neighborhood. (2) If M is a regular G-manifold and if ω is a basic form of M such that $d\omega=0$ in a neighborhood of an orbit 0, there exists a basic form $\tilde{\omega}$ of M such that $\omega-d\tilde{\omega}=0$ is zero in a neighborhood of 0. From the second theorem one can deduce a canonical isomorphism of the cohomology of the complex of basic forms of M onto the real cohomology with compact supports of M_G. *S. Chern* (Chicago, Ill.).

Gel'fand, I. M., and Graev, M. I. On a general method of decomposition of the regular representation of a Lie group into irreducible representations. Doklady Akad. Nauk SSSR (N.S.) **92**, 221–224 (1953). (Russian)

The problem treated is that of determining the value at the identity of a smooth function on a Lie group when its integrals over conjugate classes in general position are given. The groups involved are not necessarily compact, and in fact in the compact case the identity is simply a limit of conjugate classes in general position. Applying results of M. Riesz [Acta Math. **81**, 1–223 (1949); these Rev. **10**, 713] a technique is sketched for solving the problem, which is the principal difficulty in extending concrete Plancherel formulas from the case of complex to that of real Lie groups. *I. E. Segal* (New York, N. Y.).

Calabi, Lorenzo. I gruppi semisemplici di Lie che operano sullo spazio euclideo ad n dimensioni. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11** (1952), 323–335 (1953).

Let G be a connected semisimple Lie group. The author derives necessary and sufficient conditions that G can be realized as a transformation group operating transitively and effectively in euclidean space R^n. The conditions are statements about the subgroups of G, in particular about the maximal compact subgroups. The author lists explicitly all the G's which operate transitively and effectively in R^n for $n\leqq 5$. *P. A. Smith* (New York, N. Y.).

Dynkin, E. B. Homologies of compact Lie groups. Uspehi Matem. Nauk (N.S.) **8**, no. 5(57), 73–120 (1953). (Russian)

This is a general account, of the subject, from a relatively explicit, geometrical viewpoint, little use being made of recent algebraic technique. It consists, in roughly equal parts, of an account of homology theory, the application of this to compact Lie groups, and a specialization to the classical simple Lie groups. The approach leans heavily on Pontryagin's work and includes earlier work of the author [Doklady Akad. Nauk SSSR (N.S.) **85**, 697–699; **87**, 333–336 (1952); these Rev. **14**, 244, 620]. *I. E. Segal.*

Ganea, Tudor. Fortsetzung der lokalen Darstellungen topologischer Gruppen. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **3** (1951), 467–471 (1952). (Romanian. Russian and German summaries)

A topological group G is said to be locally generated by every neighborhood of the identity e if there exists a base $\beta=\{U\}$ for the neighborhoods of e with the following property: given any neighborhood N of e and any $U\in\beta$ and any $x\in U$, there exist elements $x_1, \cdots, x_n$ in U such that $x_1=e$, $x_n=x$, $x_i^{-1}x_{i+1}\in N$. Theorem: Suppose G is generated and locally generated by every neighborhood of the identity. A necessary and sufficient condition that every local homomorphism of G into an abstract group be extendible over G is that every onto-homomorphism $H\to G$ with discrete kernel (where H satisfies the same condition as G) is necessarily an isomorphism. *P. A. Smith* (New York, N. Y.).

NUMBER THEORY

Have

★Vinogradov, I. M. Osnovy teorii čisel. [Foundations of the theory of numbers.] 6th ed. Gosudarstv. Izdat. Tehn.-Teoret. Lit., Moscow, 1953. 180 pp. 5.80 rubles.

This is a reprint of the 5th edition [these Rev. **12**, 10], which was a revised form of the 4th edition [these Rev. **7**, 413]. As the reviewer of that edition observed, the main interest of the book lies in the problems, which introduce the reader to some of the author's ingenious methods of estimation, in their simplest forms. *H. Davenport.*

Kale, M. N. A note on magic squares of 9 cells. Math. Student **21** (1953), 107–108 (1954).

The number of magic squares of 9 cells whose elements are distinct positive integers in arithmetic progression with a given sum $9m$ is 8 times the greatest integer in $(m-1)/4$. *R. J. Walker* (Ithaca, N. Y.).

Palamà, Giuseppe. Su di una regola di Fermat per la fattorizzazione dei numeri e su di una sua questione relativa alle parti aliquote. Boll. Un. Mat. Ital. (3) **8**, 414–422 (1953).

Historical remarks are made about Fermat's difference-of-squares method of factorization including a number of examples where the method is extremely effective because the number to be factored has its two factors in a nearly integral ratio. The author notes that finding multiply perfect numbers requires a good factorization technique which Fermat must have had to develop for himself. *D. H. Lehmer* (Berkeley, Calif.).

Moessner, Alfred. Drei diophantische Probleme. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **8**, 191–193 (1953). (Serbo-Croatian summary)

Skolem, Th. On the diophantine equation $ax^2+by^2+cz^2=0$. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11**, 88–100 (1952).

The author repeats his proof [Norske Vid. Selsk. Forh., Trondheim **24**, 102–107 (1952); these Rev. **14**, 136] of Legendre's theorem and extends it to the case where $a=a(t)$, $b=b(t)$ and $c=c(t)$ are polynomials with integral coefficients, say of degrees n, $n-r$ and $n-s$. If r and s are not both even, non-trivial polynomial solutions $x(t)$, $y(t)$ and $z(t)$ exist if and only if $-b(t)c(t)$, $-c(t)a(t)$ and $-a(t)b(t)$ are quadratic residues modulo $a(t)$, $b(t)$ and $c(t)$ respectively in the ring of polynomials with rational coefficients. If r and s are both even, polynomial solutions exist if and only if these quadratic residue conditions are fulfilled and further a rational number t_0 exists such that $a(t_0)b(t_0)c(t_0)\neq 0$ and $a(t_0)x^2+b(t_0)y^2+c(t_0)z^2=0$ has a non-trivial solution in integers x, y, z. *I. Niven* (Eugene, Ore.).

Pompeiu, D. Une équation arithmétique. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 1–5 (1952). (Romanian. Russian and French summaries)

The problem discussed is due to Lucas [Théorie des nombres, t. 1, Gauthier-Villars, Paris, 1891] and is that of finding an integer N whose square ends in the same n digits as N itself. The paper adds nothing to what will be found in L. E. Dickson's "History of the theory of numbers" [v. 1, Carnegie Inst. Washington, 1919, pp. 458–459].
D. H. Lehmer (Berkeley, Calif.).

González-Mateo, Jesús Casanova. Calculation of the Gaussian. Gaceta Mat. (1) **5**, 147–152 (1953). (Spanish)

By the Gaussian of a modulo m is meant the least positive integer value of x for which m divides a^x-1. This is also known as the exponent of a modulo m. The author is concerned with the case of composite moduli and gives known rules for the determination of the exponent in these cases. Three simple examples are given. *D. H. Lehmer.*

Duparc, H. J. A., and van Wijngaarden, A. Note on a previous paper on Fermat's Last Theorem. Nieuw Arch. Wiskunde (3) **2**, 40–41 (1954).

In the previous paper [same Arch. (3) **1**, 123–128 (1953); these Rev. **15**, 200] the authors obtained a lower bound for z in the Fermat relation $x^n+y^n=z^n$. In the present note similar results are obtained for x and y. *D. H. Lehmer.*

Natucci, Alpinolo. Ricerche sistematiche sull'ultimo teorema di Fermat. Giorn. Mat. Battaglini (5) **1**(81), 171–179 (1953).

The author describes and illustrates a quite impractical method for looking for solutions of Fermat's equation $x^n+y^n=z^n$. Denoting $y-x$ by $\delta>0$, the author writes $d_{\delta,y}^{(n)}$ for the difference between $(y-\delta)^n+y^n$ and the nearest nth power. For each δ,n he finds the minimum absolute value with respect to y of d. If this is zero the Fermat problem is solved! *D. H. Lehmer* (Berkeley, Calif.).

Fadini, Angelo. Elementi di aritmetica nelle classi modulo n. Giorn. Mat. Battaglini (5) **1**(81), 153–170 (1953).

An expository account of the algebra of residue classes modulo n with special emphasis on the case in which n is composite. There are many illustrations. *D. H. Lehmer.*

Brenner, J. L. Linear recurrence relations. Amer. Math. Monthly **61**, 171–173 (1954).

The general recurrence relation

$$U_{n+1}=\alpha_0u_n+\cdots+\alpha_{k-1}u_{n-k+1}+\alpha,$$

where the α's are integers, is considered. By use of an interesting matrix representation the author shows that if a prime p does not divide α_{k-1} the sequence $U_0, U_1, \cdots$ is periodic modulo p, the period dividing $P=\prod_{r=0}^{k}(p^{k+1}-p^r)$. This number P is the order of a certain group of automorphisms of the additive group of vectors $(\beta_0, \beta_1, \cdots, \beta_k)$, where the β's are taken modulo p. *D. H. Lehmer.*

Ser, J. La disposition des restes dans les deux classes de nombres premiers. Mathesis **63**, 18–20 (1954).

Hagstroem, K.-G. Sulla dispersione dei numeri primi. Giorn. Ist. Ital. Attuari **14** (1951), 65–73 (1952).

Ricci, Giovanni. Errata corrige: La differenza di numeri primi consecutivi. Univ. e Politecnico Torino. Rend Sem. Mat. **12**, 315 (1953).

See same Rend. **11**, 149–200 (1952); these Rev. **14**, 727.

Linnik, Yu. V. Addition of prime numbers with powers of one and the same number. Mat. Sbornik N.S. **32**(74), 3–60 (1953). (Russian)

The author proves that every large even integer can be written as a sum of two primes and k powers of 2, where k is an absolute constant. In a previous paper [Trudy Mat. Inst. Steklov. **38**, 152–169 (1951); these Rev. **14**, 355] the author proved this theorem assuming the generalised Riemann hypothesis. If 2 is replaced by an integer $g>2$, the proof also applies, k becoming a function of g. The author also states that any large integer, written in the binary scale, can be changed into a Goldbach number by altering a bounded number of digits only. *H. Heilbronn.*

Linnik, Yu. V. Prime numbers and powers of one and the same number. Doklady Akad. Nauk SSSR (N.S.) **85**, 953–954 (1952). (Russian)

Preliminary announcement of the results in the paper reviewed above.

Mardžanišvili, K. K. On some nonlinear systems of equations in integers. Mat. Sbornik N.S. **33**(75), 639–675 (1953). (Russian)

This memoir is concerned with the solubility in positive integers $x_1, \cdots, x_s$ of the system of equations

$$(1)\qquad x_1^k+\cdots+x_s^k=N_k \quad (k=l, m, \cdots, n),$$

where $l, m, \cdots, n$ are g distinct positive integers of which n is the greatest. The treatment represents a generalisation of Vinogradov's work [Trav. Inst. Math. Stekloff **23** (1947); these Rev. **10**, 599] on Waring's problem, which is the case $g=1$. Some condition on the relative magnitudes of N_l, $N_m, \cdots, N_n$ is obviously necessary. The author puts $N_k=h_k(N_n)^{k/n}$ for $k=l, m, \cdots, n$, and postulates that the equations $\xi_1^k+\cdots+\xi_g^k=h_k$ $(k=l, m, \cdots, n)$ have a solution in real $\xi_1, \cdots, \xi_g$ for which $\xi_1, \cdots, \xi_g$ and $|\det(\xi_t^{k-1})|$, where $t=1, \cdots, g$, all have a fixed positive lower bound. Suppose $n\geqq 12$ and $f>3ng$, and put

$$r=[2n \log (10ng)+n \log\log (20ng)]+1.$$

In the first part of the paper the author obtains a lower bound for the number of solutions of (1) when $s=f+2gr$. The details of the work are necessarily heavy, as the single integral occurring in the Hardy-Littlewood and Vinogradov work is replaced here by integration over g variables. The significance of the result depends naturally on whether the "singular series" for the problem, which occurs as a factor in the main term of the lower bound, is strictly positive, and this question, which is purely arithmetical, is investigated in the second part of the paper. It is shown that if certain congruences are soluble and if s is greater than a certain number depending on g and on the primes $\leqq n^{n+1}$, then the desired property holds. *H. Davenport.*

Nečaev, V. I. On the representation of natural numbers as a sum of terms of the form

$$\frac{x(x+1)\cdots(x+n-1)}{n!}.$$

Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 485–498 (1953). (Russian)

Let $\varphi_n(x)$ denote the polynomial of the title, and let $g(\varphi_n)$ denote the least r with the property that every positive integer is representable as a sum of at most r values of $\varphi_n(x)$, arising from non-negative integral values of x. In his monograph [Trudy Mat. Inst. Steklov. **38**, 190–243

(1951); these Rev. 13, 914] the author proved that

$$g(\varphi_n) < \tfrac{1}{2}n^2 \log n + 6n \log n$$

for $n \geqq 12$. In the present paper he improves the right-hand side to $6n \log n + 9n \log \log n$. The proof is based on the work of the monograph, but requires improved estimates at many points. The main tool by which these are obtained is the use of the inequality

$$\left|\sum_{x=1}^{p} \exp\,(2\pi i f(x)/p)\right| < np^{1/2}$$

for exponential sums (p prime, $f(x)$ a polynomial of degree n), due to A. Weil [Proc. Nat. Acad. Sci. U. S. A. 34, 204–207 (1948); these Rev. 10, 234]. Use is also made of some of Tchebychev's estimates for the distribution of primes.

H. Davenport (London).

Rieger, Georg Johann. Zur Hilbertschen Lösung des Waringschen Problems: Abschätzung von $g(n)$. Mitt. Math. Sem. Giessen no. 44, 35 pp. (1953).

The results of this dissertation have been summarized in a note which appeared in Arch. Math. 4, 275–281 (1953); these Rev. 15, 289. *R. D. James* (Vancouver, B. V.).

Richert, Hans-Egon. Über quadratfreie Zahlen mit genau r Primfaktoren in einer arithmetischen Progression. J. Reine Angew. Math. 192, 180–203 (1953).

Consider the square-free numbers with exactly r factors and let $\pi_r(x; k, l)$ denote the number of such numbers $\leqq x$ that are congruent to l (mod k). The first main result of the paper is an asymptotic formula for $\pi_r(x; k, l)$ in which the error term is uniform in k. The result is too elaborate to reproduce here. For $r=1$ it reduces to the well known theorem of Page-Siegel-Walfisz. As an application of this result the writer obtains an asymptotic formula for the number of solutions of $n=p_1+p_2+p_3p_4$, $p_3<p_4$, where the p_i are primes. This includes a theorem of Estermann [Quart. J. Math., Oxford Ser. 8, 32–38 (1937)].

L. Carlitz (Durham, N. C.).

Cugiani, Marco. Sui valori di un polinomio che risultano liberi da potenze. Ann. Mat. Pura Appl. (4) 35, 291–298 (1953).

Let g be a fixed integer greater than unity. The author has previously shown [same Ann. (4) 33, 135–143 (1952); these Rev. 14, 356] that every sufficiently large integer N can be represented in the form $N=x^g+l_g$, where x is a positive integer and l_g is a gth-power-free integer. In the present paper he proves that if $E(N)$ denotes the number of such representations, then

$$E(N) \sim N^{1/g} \prod_{p^g \mid N}\left(1-\frac{1}{p}\right) \prod_{p \nmid N}\left(1-\frac{\nu_N(p^g)}{p^g}\right),$$

where $\nu_N(m)$ is the number of solutions of the congruence $x^g \equiv N \pmod m$. In the case $g=2$ this result reduces to a formula of Estermann [Math. Ann. 105, 653–662 (1931)]. The author also proves an analogous theorem for the function $E'(N)$ in which the representations involved are subject to the additional restriction that $(x, N)=1$. The main tool employed is the method of Viggo Brun.

A. L. Whiteman (Princeton, N. J.).

Pfetzer, Werner. Die Wirkung der Modulsubstitutionen auf mehrfache Thetareihen zu quadratischen Formen ungerader Variablenzahl. Arch. Math. 4, 448–454 (1953).

The author applies a method of Hecke [Math. Ann. 97, 210–242 (1926)] to the problem of determining the level (Stufe) of the modular form given by the theta series based on a positive definite quadratic form (with rational integral coefficients) in n variables, where n is odd, and to a more general series obtained by applying powers of certain linear first-order differential operators in n variables. For an even number of variables, the analogous discussion was given by B. Schoeneberg [ibid. 116, 511–523 (1939)]. The present paper obtains most of Schoeneberg's results, the principal one being that the class of modular forms has in all cases level $2N$, in some cases level N, where $N=D/K$, D being the determinant of the quadratic form, and K the greatest common divisor of the $(n-1)$-rowed determinants formed from the matrix of the quadratic form. *J. Lehner.*

Koecher, Max. Zur Theorie der Modulformen n-ten Grades. I. Math. Z. 59, 399–416 (1954).

Siegel's definition [Math. Ann. 116, 617–657 (1939); these Rev. 1, 203] of modular forms of degree n includes restrictions on the behavior of the forms at infinity. The principal purpose of the present paper is to show that these restrictions can be dropped for $n>1$. Regularity and the existence of suitable transformation formulas are enough to insure proper behavior at infinity. This is proved for certain subgroups of the symplectic group as well as for the modular group of degree n. The corresponding field of modular functions is also considered. *H. S. Zuckerman.*

Rankin, R. A., and Rushforth, J. M. The coefficients of certain integral modular forms. Proc. Cambridge Philos. Soc. 50, 305–308 (1954).

If k is even, $k \geqq 12$, then the space of cusp-forms of dimension $-k$ belonging to the full modular group has a finite basis $f_k^{(j)}(z)=\sum_{n=1}^{\infty}\lambda_k^{(j)}(n)e^{2\pi i n z}$. The real algebraic numbers $\lambda_k^{(j)}(n)$ are shown to be algebraic integers belonging to a field $K_k^{(j)}$. Bounds for the degree of $K_k^{(j)}$ over the field of rational numbers are given. *H. S. Zuckerman.*

Carlitz, L. A note on modular invariants. Nieuw Arch. Wiskunde (3) 2, 28–31 (1954).

Consider a system of forms $f_1, \cdots, f_r$ with coefficients in GF(q) which are subjected to the transformations of the full linear group with coefficients in GF(q). Let the classes of equivalent systems be denoted by $C_1, \cdots, C_k$ and let any particular invariant V take the value v_i for the class C_i. Now assume that the numbers v_i belong to a field $\Phi = \mathrm{GF}(q_1)$ which contains k distinct numbers $\alpha_1, \cdots, \alpha_k$ and define the invariant J by means of $J(C_i)=\alpha_i$. Theorem 1. Every invariant V can be exhibited as a polynomial in J of degree $\leqq k-1$ with coefficients in Φ. Theorem 2. There exist s invariants $J_1, \cdots, J_s$, where s satisfies $q_1^{s-1}<k\leqq q_1^s$, such that an arbitrary invariant V can be exhibited as a polynomial in the J's of degree $\leqq q_1-1$ in each J and with coefficients in GF(q_1). Moreover, the number s cannot be diminished. *A. L. Whiteman* (Princeton, N. J.).

Carlitz, L. A note on Wolstenholme's theorem. Amer. Math. Monthly 61, 174–176 (1954).

Put $S_{k,m}=\sum(km+r)^{-1}$, where k is an arbitrary integer, and the summation is restricted to integers r in the range $1\leqq r\leqq m$ with r prime to m. Generalizing the well-known

theorems of Wolstenholme and Leudesdorf, the author proves: (1) If m is equal to a prime p, then $S_{k,p}\equiv 0 \pmod{p^{s+2}}$ provided $p^s|(2k+1)$, $p>3$; also $S_{k,p}\equiv 0 \pmod{p^{s+1}}$ for $p=3$. (2) If m is an integer relatively prime to 6, then $S_{k,m}\equiv 0 \pmod{(2k+1)m^2}$. *A. L. Whiteman* (Princeton, N. J.).

Carlitz, Leonard. The coefficients of singular elliptic functions. Math. Ann. **127**, 162–169 (1954).

Let $\operatorname{sn} x = \operatorname{sn}(x, u)$ denote the Jacobi elliptic function, where $u=k^2$ in the usual notation. Then

$$\operatorname{sn} x = \sum_{m=0}^{\infty} A_{2m+1}(u) \frac{x^{2m+1}}{(2m+1)!} \quad (A_1(u)=1),$$

where the $A_{2m+1}(u)$ are polynomials in u with integral coefficients. Put $x/\operatorname{sn} x = \sum_{m=0}^{\infty} \beta_{2m}(u) x^{2m}/(2m)!$ $(\beta_0(u)=1)$, so that the $\beta_{2m}(u)$ are polynomials in u with rational coefficients. The author supposes that the function sn x admits of complex multiplication, so that the period quotient belongs to an imaginary quadratic field of discriminant d. If k^2 is the corresponding singular modulus, he writes $\alpha_m = A_m(k^2)$, $\beta_m=\beta_m(k^2)$, $\tau_m=\beta_m/m$. Let p be an odd prime such that the Legendre symbol $(d/p)=-1$. Then the author shows that: (1) $\alpha_p \equiv O \pmod p$, so that β_m is now integral (mod p) for all m; (2) $\alpha_m\equiv O \pmod{p^r}$ for $m\geqq pr$; (3) $\tau_m \equiv O \pmod{p^r}$ for $m>pr$, $p^2-1\nmid m$; (4) $\beta_m\equiv O \pmod{p^{r-1}}$ for $m>pr$, $p^2-1|m$, $p\nmid m$. In conclusion the author derives an auxiliary congruence involving the function

$$\sigma_s(u) = \beta_{s(p-1)}(u) + (p^{-1}-1) A_p{}^s(u).$$

A. L. Whiteman (Princeton, N. J.).

Carlitz, Leonard. Congruence properties of the polynomials of Hermite, Laguerre and Legendre. Math. Z. **59**, 474–483 (1954).

In this paper the author derives congruence properties of the classical orthogonal polynomials. The main results are as follows. (1) For the Hermite polynomial $H_n(x)$ we have $H_{n+m}(x)\equiv(2x)^m H_n(x) \pmod m$, where m is an arbitrary integer. (2) Let $L_n^{(\alpha)}(x)$ denote the Laguerre polynomial of degree n and put $\Lambda_n^{(\alpha)}(x)=n!L_n^{(\alpha)}(x)$. Then

$$\Lambda_{n+m}^{(\alpha)}(x) \equiv \Lambda_m^{(\alpha)}(x)\Lambda_n^{(\alpha)}(x) \pmod m,$$

where the parameter α is an indeterminate or a rational number that is integral (mod m). (3) For the Legendre polynomial $P_n(x)$ we have $n!P_{n+m}(x)\equiv n!P_n(x)P_m(x) \pmod m$ and $n!P_{m-n-1}(x)\equiv n!P_n(x)P_{m-1}(x) \pmod m$ for $m>n$, where m is an arbitrary odd integer. The results for the Hermite and Laguerre polynomials are proved in a uniform manner by means of a theorem on difference equations of the second order. *A. L. Whiteman* (Princeton, N. J.).

Carlitz, L. Representations by quadratic forms in a finite field. Duke Math. J. **21**, 123–137 (1954).

Let $q=p^n$, $p>2$, and let $A(x)$, $B(x)$ be two quadratic forms with coefficients in $GF(q)$, with m and t variables, and with corresponding matrices A and B of rank m and r respectively $(r\leqq m, t)$. The author seeks the number $N_t(A, B)$ of $m\times t$ matrices X with elements in $GF(q)$ such that $X'AX=B$. For $\alpha \in GF(q)$ define $\psi(\alpha)=0, 1, -1$ according as $\alpha=0$, square or non-square of $GF(q)$. Let δ_B equal the product of the r non-zero elements in the diagonal form of B and put $\lambda_B=\psi(\delta_B)$. Theorem 1 is a simple product formula which exhibits $N_r(A, B)$ as a function of λ_A, λ_B, m and r. Theorem 3 gives the number $N(m, r, \lambda)$ of symmetric matrices A of order m, rank r and such that $\lambda_A=\lambda$. Theorem 4 expresses $N_t(A, 0)$ in terms of certain terminating q-hypergeometric series. Theorem 5 is a formula for computing $N_t(A, B)$ when the formulas for $N_r(A, B)$ and $N_t(A, 0)$ are given. The author discusses also a certain sum $H(B, r, \lambda)$ which for $B=0$ reduces to $N(M, r, \lambda)$. Finally he considers applications to the problem of representing a polynomial in $GF[q, x]$ as a sum of squares.

A. L. Whiteman (Princeton, N. J.).

Lewis, D. J. Singular quartic forms. Duke Math. J. **21**, 39–44 (1954).

By a singular zero of a form (homogeneous polynomial) over a finite field k is meant a common zero of the form and its formal partial derivatives; a singular form is a form for which all zeros lying in the field of coefficients of the form are singular. Quadratic and cubic forms are disposed of readily. For the quartic case it is proved that if k is sufficiently large and not of characteristic 2 then the quartic is of one of the following types. I. $\epsilon(Q_1^2-Q_2^2)$, $\epsilon=\pm1$, ν non-ϵ k^2. II. $\epsilon(L_1^2-\nu L_2^2)(L_3^2-\mu L_4^2)$, $\epsilon=\pm1$, ν, μ non-ϵ k^2. III. $U(L_1, \cdots, L_t)$, where U is a quartic form with only the trivial zero in k so that $t\leqq 4$. Here Q and L denote quadratic and linear forms, respectively. The proof of the theorem depends on an extension of a theorem of the reviewer [same J. **19**, 471–474 (1952); these Rev. **14**, 539]. *L. Carlitz.*

Parodi, Maurice. Un critère d'irréductibilité des polynomes à coefficients entiers sur le corps des nombres rationnels. C. R. Acad. Sci. Paris **237**, 1057–1059 (1953).

Let $f(x)=x^n+a_2x^{n-2}+a_3x^{n-3}+\cdots+a_n$ be a polynomial with integral rational coefficients and $a_n\neq0$. Assume that $|a_2|>1+|a_3|+|a_4|+\cdots+|a_n|$. It was proved by the reviewer that $f(x)$ is irreducible in the field of rational numbers if $a_2>0$ [Amer. J. Math. **70**, 423–432 (1948); these Rev. **10**, 14]. Using theorems on characteristic roots of matrices, the author gives a new simple proof of this result. Moreover, he proves that $f(x)$ is also irreducible if $a_2<0$ provided that $f(x)$ does not change its sign in two given intervals. *A. Brauer* (Chapel Hill, N. C.).

Günther, Alfred. Über transzendente $\mathfrak{p}$-adische Zahlen. I. Ein Satz über algebraische Abhängigkeit $\mathfrak{p}$-adischer Funktionen als Prinzip für Transzendenzbeweise $\mathfrak{p}$-adischer Zahlen. J. Reine Angew. Math. **192**, 155–166 (1953).

A $\mathfrak{p}$-adic analog of a theorem of T. Schneider [Math. Ann. **121**, 131–140 (1949); these Rev. **11**, 160] concerning the algebraic dependence of a set of integral-valued functions is established. From this the author obtains the two following $\mathfrak{p}$-adic generalizations of two theorems of Mahler [J. Reine Angew. Math. **169**, 61–66 (1932); Compositio Math. **2**, 259–275 (1935)] and a theorem of Veldkamp [J. London Math. Soc. **15**, 183–192 (1940); these Rev. **2**, 149]: Let p be the rational prime number of which $\mathfrak{p}$ is a factor and put $p^{-1/(p-1)}=q$. 1. Let α be a $\mathfrak{p}$-adic number such that $0<|\alpha|_\mathfrak{p}<q$; then one at least of the two numbers α and e^α is transcendental. 2. Let $\alpha\neq0, 1$ be such that $|\alpha-1|_\mathfrak{p}<q$ and β an irrational $\mathfrak{p}$-adic integer; then one at least of the numbers α, β, α^β is transcendental. It is further shown that for a $\mathfrak{p}$-adic number α with $0<|\alpha|_\mathfrak{p}<q$ one at least of the numbers α and $\sin\alpha$ (or $\cos\alpha$) is transcendental.

O. Taussky-Todd (Washington, D. C.).

Dufresnoy, J., et Pisot, Ch. Sur un ensemble fermé d'entiers algébriques. Ann. Sci. Ecole Norm. Sup. (3) 70, 105–133 (1953).

Dufresnoy, J., et Pisot, Ch. Sur les dérivés successifs d'un ensemble fermé d'entiers algébriques. Bull. Sci. Math. (2) 77, 129–136 (1953).

The first paper contains detailed proofs of results which have been stated in two earlier notes [C. R. Acad. Sci. Paris **235**, 1592–1593 (1952); **236**, 30–31 (1953); these Rev. **14**, 538]. Let S be the set of algebraic integers $\theta>1$ such that all its conjugates (except θ itself) lie in the interior of the unit circle $|z|<1$. It is known [Salem, Duke Math. J. **11**, 103–108 (1944); **12**, 153–172 (1945); these Rev. **5**, 254; **6**, 206] that the set S is closed and has non-empty derived sets of any finite order. C. L. Siegel [ibid. **11**, 597–602 (1944); these Rev. **6**, 39] has found the two smallest numbers of S and has conjectured that the smallest number of the derived set S' is $\frac{1}{2}(1+\sqrt{5})$.

The authors prove: 1) the necessary and sufficient condition for a number $\theta \in S$ to belong to S' is the existence of a polynomial $A(z)$ with integral coefficients such that, for $|z|=1$, the inequality $|A(z)|\leqq|P(z)|$ should be verified (P being the irreducible polynomial with integral coefficients and with root θ), the equality $|A(z)|=|P(z)|$ holding only at a finite number of points; 2) if $\theta \in S$ and $n\geqq 2$, then $\theta^n \in S'$; 3) if $\theta \in S$ and θ is totally real, then $\theta \in S'$; 4) $\frac{1}{2}(1+\sqrt{5})$ is, according to Siegel's conjecture, the smallest number of S'. They also determine the four smallest elements of the set S.

In the second paper the authors study the derived sets $S^{(n)}$ of order n of the set S and prove that S does not possess a nonempty derived set of transfinite order. *R. Salem.*

Masuda, K. Note on conservative algebraic function fields. Tôhoku Math. J. (2) 5, 12–17 (1953).

Let k be a separably generated function field in one variable over a constant field k_0 of characteristic $p\neq 0$, and let x be a separating variable. k is called conservative if the genus remains the same in all constant field extensions of k. Masuda proves that k is conservative if and only if all of the prime ideals of $k_0(x)$ which correspond to irreducible polynomials of the form $f(x^p) \in k_0[x]$ are unramified in k. The proof is based on the following lemma. Let S_1 and S_2 be two finite extensions of a field S. Let $\Sigma_1, \Sigma_2, \Sigma_{12}$ be the rings of integers in S_1, S_2 and in the compositum S_1S_2, with respect to some discrete valuation of S_1S_2, and let e_1 and e_2 be the ramification indices of S_1 and S_2 over S. Then, if there exists a non-unit in Σ_{12} with an order smaller than $\min(1/e_1, 1/e_2)$, we have $\Sigma_{12}\neq\Sigma_1\Sigma_2$. *J. T. Tate.*

Jaffard, Paul. Extensions algébriques infinies de PF-corps. Ann. Sci. Ecole Norm. Sup. (3) 70, 181–198 (1953).

Let k be a field and let $\{\varphi_\iota\ (\iota \in I)\}$ be a family of (non-trivial and mutually inequivalent) valuations on k. Let k_0 be the set of all elements x in k such that $\varphi_\iota(x)\leqq 1$ for all $\iota \in I$. For a non-archimedian φ_ι, let $\bar{k}_{\varphi_\iota}$ be the associated residue-field. Artin [Algebraic numbers and algebraic functions, v. I, Inst. Math. Mech., New York Univ., 1951; these Rev. **13**, 628], and Artin and Whaples [Bull. Amer. Math. Soc. **51**, 469–492 (1945); **52**, 245–247 (1946); these Rev. **7**, 111, 410] have considered the following axioms: 1) For each $x\neq 0$ in k, $\varphi_\iota(x)=1$ for almost all $\iota \in I$ and $\prod_\iota \varphi_\iota(x)=1$; 2) among φ_ι there is (at least) a reasonable one, i.e. a φ_ι such that either φ_ι is archimedian, or φ_ι is discrete-archimedian and $\bar{k}_{\varphi_\iota}$ is finite, or (there is no archimedian valuation in our family whence k_0 is a subfield of k algebraically closed in k and) φ_ι is discrete and $\bar{k}_{\varphi_\iota}$ is finite over k_0. They have characterized algebraic number fields of finite degree and algebraic function fields of finite degree of one variable as fields with a family $\{\varphi_\iota\}$ satisfying these axioms 1 and 2, i.e. as PF-fields. The present paper studies algebraic extensions, not necessarily of finite degree, of PF-fields and gives their axiomatic characterization. Thus let Ω be an algebraic extension of a PF-field k, and $\mathfrak{D}_\Omega$ be the set of all places (equivalently, valuations) in Ω which are extensions of those in k appearing in the product formula of k. For a subfield K of Ω containing and finite over k, let $\mathcal{E}_K$ be the totality of pairs $(\mathfrak{P}, \mathfrak{Q})$ $(\mathfrak{P}, \mathfrak{Q} \in \mathfrak{D}_\Omega)$ such that $\mathfrak{P}, \mathfrak{Q}$ have the same restriction on K, and let $K(\mathfrak{P})$ be the totality of $\mathfrak{Q} \in \mathfrak{D}_\Omega$ with $(\mathfrak{P}, \mathfrak{Q}) \in \mathcal{E}_K$. A uniform structure is introduced in $\mathfrak{D}_\Omega$ taking the $\mathcal{E}_K$ with varying K as uniform neighborhoods, and in this sense $\mathfrak{D}_\Omega$ is shown to be complete, and indeed locally compact.

On $\mathfrak{D}_\Omega$ is defined a positive Radon measure μ which extends a certain linear function, explicitly given by means of "ramification indices" and "norms", on the space of those functions on $\mathfrak{D}_\Omega$ which are constant on each $K(\mathfrak{P})$ and are 0 on almost all $K(\mathfrak{P})$, for a fixed subfield K as above. Then for each $x\neq 0$ in Ω the function $\bar{x}$: $\mathfrak{P}\to\varphi_\mathfrak{P}(x)$ on $\mathfrak{D}_\Omega$ is continuous and is 0 except on a compact set of null measure, where $\varphi_\mathfrak{P}$ is the valuation associated with the place $\mathfrak{P}$. Conversely, a field Ω is an algebraic extension of a PF-field when it satisfies the following three axioms: 1′) There is a locally compact set $\mathfrak{D}_\Omega=\{v\}$ of valuations of Ω with a positive measure μ and for every $x\neq 0$ in Ω the function $\bar{x}$: $v\to v(x)$ on $\mathfrak{D}_\Omega$ is continuous and has a compact carrier of measure 0; 2′) for every subfield K of Ω generated over R by a single element, there is a valuation v in $\mathfrak{D}_\Omega$ such that its restriction v_K to K is reasonable, where R is the prime field if $\mathfrak{D}_\Omega$ contains an archimedian valuation while R is a subfield of Ω obtained by the adjunction of a single transcendental element to the field k_0 consisting of all elements x such that $v(x)\geqq 0$ for every $v \in \mathfrak{D}_\Omega$; 3) for each K and for every $v \in \mathfrak{D}_\Omega$ the subset $K(v)$ of $\mathfrak{D}_\Omega$ is open. A modified form of the axiom 2′ is introduced. Also a modification of the axiom 1′ is given, which corresponds to the weaker product formula in the second of the cited papers by Artin and Whaples. Ω being again an algebraic extension of a PF-field k, let ξ be a continuous function of compact carrier and set $S=\{x \in \Omega \mid x=0$ or $v(x)\geqq\xi(v)$ for all $v \in \mathfrak{D}_\Omega\}$. It is proved that there is a constant l such that, for every subfield K of Ω containing and finite over k, the order M of $S\cap K$, in the sense of Artin [loc. cit.] satisfies $M\leqq l^{[K:k]}\mathfrak{B}(\xi)^{[K:k]}$ where

$$\mathfrak{B}(\xi)=\exp\left(-\int \xi\, d\mu\right).$$

T. Nakayama (Nagoya).

Kubota, Tomio. Über die Beziehung der Klassenzahlen der Unterkörper des bizyklischen biquadratischen Zahlkörpers. Nagoya Math. J. 6, 119–127 (1953).

Let K be a field normal of degree four over the field of rational numbers. If K is not cyclic, it has three quadratic subfields k_0, k_1, k_2. If h_i denotes the class number of k_i and h that of K, one has $4h=Qh_0h_1h_2$ if K is real and $2h=Qh_0h_1h_2$ if K is non-real. Here, Q denotes the index of the group generated by the units of k_0, k_1, k_2 in the group of all units of K. This is an extension of a theorem of Dirichlet which was obtained by G. Herglotz [Math. Z. **12**, 255–261 (1922)]. The author gives a proof which does not make use of the zeta-function. *R. Brauer* (Cambridge, Mass.).

Whaples, G. Existence of generalized local class fields. Proc. Nat. Acad. Sci. U. S. A. **39**, 1100–1103 (1953).

Generalized local class-field theory deals with the extensions of a field k complete with respect to a discrete valuation whose residue class field is not necessarily finite, but is perfect, and has exactly one extension of degree n for each natural number n. Whaples proves that a subgroup a of k^* is the norm subgroup of a finite abelian extension of k, if and only if a is of finite index, has a conductor, and satisfies the following condition: For each positive integer i there exists a polynomial $u_i(x)$ with integral coefficients in k, having positive degree when reduced modulo the prime, such that $1+u_i(\xi)\pi^i \in a$ for all integers ξ of k, π being a fixed prime element. *J. T. Tate* (New York, N. Y.).

Takahashi, Shuichi. Homology groups in class field theory. Tôhoku Math. J. (2) **5**, 8–11 (1953).

The reviewer has shown that the Galois cohomology group $H^r(G, A)$ of the idèle class group A of an algebraic number field is canonically isomorphic to the integral cohomology group $H^{r-2}(G, Z)$ for $r>2$, and has stated that the same holds for all integers r, if one introduces appropriate negative-dimensional cohomology groups [Ann. of Math. (2) **56**, 294–297 (1952); these Rev. **14**, 252]. Takahashi observes that the homology groups of G in A will serve as the negative-dimensional cohomology groups in question, if we put $H^{-r}(G, A)=H_{r-1}(G, A)$ for $r>1$. Then, for $r<0$, the canonical isomorphism $H^{r-2}(G, Z)\simeq H^r(G, A)$ is given by the cap product with the fundamental 2-cocycle, and the proof is the same as in the case $r>0$. Finally, it is shown that the special case $r=-1$ of this isomorphism is a generalization to non-abelian G, of the isomorphism of H. Kuniyoshi in T. Tannaka's theory of the Hauptgeschlechtsatz in Minimalen. *J. T. Tate* (New York, N. Y.).

Tamagawa, Tsuneo. On the theory of ramification groups and conductors. Jap. J. Math. **21** (1951), 197–215 (1952).

Let k be an algebraic number field and let K/k be a normal extension with group $\mathfrak{G}$. Let C_K be the idèle class group of K, and let $G(K, k)$ be the Weil group of K/k, that is, the group extension of C_K by $\mathfrak{G}$ belonging to the fundamental class in $H^2(\mathfrak{G}, C_K)$. Weil has attached an L-series to each (non-abelian) character χ of $G(K, k)$, obtaining in this way a common generalization of Artin's non-abelian L-series and of Hecke's L-series with Grössencharactere. Tamagawa makes the corresponding generalization of the theory of conductors. For each non-archimedian prime $\mathfrak{p}$ of k, the local Weil group $G(K_\mathfrak{P}, k_\mathfrak{p})$ can be imbedded in the global one, and by restriction, χ induces a character $\chi_\mathfrak{p}$ of $G(K_\mathfrak{P}, k_\mathfrak{p})$ which is independent of the choice of imbedding. Now $G(K_\mathfrak{P}, k_\mathfrak{p})$ is essentially the same as the Galois group of $A_\mathfrak{P}$ over $k_\mathfrak{p}$, where $A_\mathfrak{P}$ is the maximal abelian extension of $K_\mathfrak{P}$; and by using the ramification groups of this infinite extension, whose definition and properties he develops in considerable detail, Tamagawa attaches an exponent $e(\mathfrak{p})$ to the local character $\chi_\mathfrak{p}$ by means of Artin's well known formula. Defining $f(\chi, K/k)=\prod \mathfrak{p}^{e(\mathfrak{p})}$, he obtains conductors which have all of the usual formal properties, and which are equal to the old conductors in the special cases of Artin and Hecke L-series. *J. T. Tate* (New York, N. Y.).

Kawada, Yukiyosi. On the ramification theory of infinite algebraic extensions. Ann. of Math. (2) **58**, 24–47 (1953).

In editing a posthumous paper of J. Herbrand [Math. Ann. **108**, 699–717 (1933)], C. Chevalley showed that the naive definition of higher ramification groups is unsatisfactory for infinite extension fields and developed a theory of ramification groups in infinite extensions as projective limits of those of the finite subfields. He did not give an explicit parametrization for these ramification groups. This paper uses the function $\varphi_{K/k}$, introduced by Hasse [J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. **2**, 477–498 (1934)] for finite normal K/k, where k is a p-adic number field. This function is defined for real argument $\geqq -1$, has a convex graph consisting of a finite set of straight lines, and had been implicitly used by Herbrand [J. Math. Pures Appl. (9) **10**, 481–498 (1931)] to describe the relation between the ramification groups of a (finite) extension field and those of an intermediate field. [See also the paper reviewed above and Satake, Sci. Papers Coll. Gen. Ed. Univ. Tokyo **2**, 25–39 (1952); these Rev. **14**, 452.] Let k be a field complete under a discrete valuation with perfect residue class field. Let K/k be finite normal separable. For each real $u\geqq -1$ the author defines $\mathfrak{U}_{K/k}(u)=\mathfrak{B}_{K/k}^{(i)}$ where i is the least integer $\geqq \varphi_{K/k}(u)$, and $\mathfrak{U}^*_{K/k}(u)=\mathfrak{B}_{K/k}^{(j)}$ where j is the least integer $>\varphi_{K/k}(u)$. When K/k is infinite normal separable, $\mathfrak{U}_{K/k}(u)$ and $\mathfrak{U}^*_{K/k}(u)$ can now be defined by projective limits, thus parametrizing the set of ramification groups defined by Chevalley. The author then studies the sets of groups $\{\mathfrak{U}_{K/k}(u)\}$ and $\{\mathfrak{U}^*_{K/k}(u)\}$, giving examples of the various situations which can arise, and defines conductor and different for infinite extensions. *G. Whaples* (Bloomington, Ind.).

Fischer, Wilhelm. Über die Zetafunktion des reellquadratischen Zahlkörpers. Math. Z. **57**, 94–115 (1952).

The author considers the Dedekind zeta-function $\zeta(s, \mathsf{K})$ of an ideal class K of a real quadratic field and proves for it an analogue of the approximate functional equation which Hardy and Littlewood proved for the Riemann zeta-function. The proof follows roughly the lines of Hardy and Littlewood's second proof of their relation [Proc. London Math. Soc. (2) **21**, 39–74 (1922)], which in turn was modeled after Riemann's first proof of the exact functional equation for the Riemann zeta-function. However, certain difficulties arise in carrying over the Hardy-Littlewood proof to the present situation. For example, it seems necessary to obtain first an approximate functional equation involving both $\zeta(s, \mathsf{K})$ and $\zeta(s-1, \mathsf{K})$ and to obtain the desired approximate functional equation for $\zeta(s, \mathsf{K})$ by a differencing process. *P. T. Bateman* (Urbana, Ill.).

Denjoy, Arnaud. L'équation fonctionnelle de $\zeta(s)$. C. R. Acad. Sci. Paris **238**, 533–536 (1954).

Still another proof of the functional equation for the Riemann zeta-function. The usual formula $\zeta(s)=\sum_1^\infty m^{-s}$ $(\sigma>1)$ leads to

$$\zeta(s)=\lim_{n\to\infty}\left\{\sum_1^n m^{-s}+(n+\tfrac{1}{2})^{1-s}/(s-1)\right\} \quad (\sigma>-1).$$

The result follows from this and transformations of the formula

$$\Gamma(s)\zeta(s)=\int_0^\infty \frac{x^{s-1}}{e^x-1}dx.$$

E. C. Titchmarsh (Oxford).

Dénes, Peter. Über die Kummerschen logarithmischen Hilfsfunktionen. Acta Sci. Math. Szeged **15**, 115–125 (1954).

Let l denote an odd prime, ζ a primitive lth root of unity, $R(\zeta)$ the cyclotomic field generated by ζ, $\lambda=1-\zeta$, $L=(\lambda)$,

so that $(l)=L^{l-1}$. Let $\omega=\sum_{i=0}^{s}a_i\zeta^i$ denote an integer of $R(\zeta)$: thus for each ω there is an expression of this sort with $s\leqq l-2$, which may be called the normal form and is denoted by ω^*. Together with ω we consider $\omega(e^v)=\sum_0^s a_i e^{iv}$. In a previous paper [Publ. Math. Debrecen 2, 206–214 (1952); these Rev. 14, 728] the author proved the congruence

$$D_k \log \omega_1(e^v)\equiv D_k \log \omega_2(e^v) \pmod{l}$$

for $k=1, \cdots, l-2$, which had been stated by Kummer; here $\omega_1=\omega_2$ and is prime to L, and D_k indicates the kth derivate with respect to v at $v=0$. The main results of the present paper are the following.

I. Let $(\omega, L)=1$ and let

$$D_{il^y} \log \omega(e^v)\equiv 0 \pmod{L^{y+1}} \qquad (y=1, \cdots, n-1; i=1, \cdots, l-2),$$
$$D_{kl^n} \log \omega(e^v)\equiv 0 \pmod{L^{n+1}} \quad (k=1, \cdots, t-1),$$
$$D_{tl^n} \log \omega(e^v)\equiv q\not\equiv 0 \pmod{L^{n+1}} \quad (0<t<l-1);$$

then

$$\omega\equiv f_0+f_t\lambda^{n(l-1)+t} \pmod{L^{n(l-1)+t+1}},$$

where f_0 and f_t are rational integers prime to l and

$$f_0\equiv\omega_0+\frac{l}{l-1}D_{l^n(l-1)}\omega(e^v) \pmod{l^{n+1}},$$

$$f_t\equiv(-1)^t\omega_0 q/(l^n\cdot t!) \pmod{l}.$$

II. Let $(\omega, L)=1$ and let ω be congruent to a rational integer $\pmod{L^n}$; then $\omega_0\equiv\omega_0^* \pmod{l^{n+1}}$ if and only if $D_{l^n(l-1)} \log \omega(e^v)\equiv 0 \pmod{L^n}$. *L. Carlitz.*

Selberg, Sigmund. On a conjecture by Ernst Jacobsthal. Norske Vid. Selsk. Forh., Trondheim 26 (1953), 89–93 (1954).

The author shows that the expression

$$\sigma_k(x)=\sum_{l=1}^{k}\left\{x\left[\frac{l}{x}\right]-(x+1)\left[\frac{l}{x+1}\right]\right\},$$

which occurs in some work of Jacobsthal, is always non-negative for $x>0$. *R. Bellman* (Santa Monica, Calif.).

Oppenheim, A. One-sided inequalities for quadratic forms. II. Quaternary forms. Proc. London Math. Soc. (3) 3, 417–429 (1953).

[For part I see same Proc. (3) 3, 328–337 (1953); these Rev. 15, 291.] Let $f(x_1, \cdots, x_n)$ be an indefinite quadratic form in integral variables and nonzero determinant $\Delta(f)$ with signature s. Let $P_1(f)$ denote the lower bound of the positive values of f; let $P_2(f)=P_1(-f)$ and $J_1=P_1^n/|\Delta|$, $J_2=P_2^n/|\Delta|$. Davenport [same Proc. (2) 51, 145–160 (1949); these Rev. 10, 593] and the author (not published yet) deduced estimates for J_1 and J_2 in the ternary case. Now the author gives similar results in the quaternary case: If $n=4$, $s=0$, $\Delta(f)>0$, then either $J_1\leqq 256/81$, or else $J_1=16$ and f is $P_1(xy+zt)$, or $J_1=81/16$ and f is $\frac{1}{3}P_1(-x^2+4y^2+4zt)$, or $J_1=4$ and f is $P_1(xy+2zt)$ or $P_1(xy+z^2-t^2)$, or $J_1=16/5$ and f is $P_1(xy+z^2+zt-t^2)$ (the same for P_2). If $n=4$, $s=2$, $\Delta(f)<0$, then (a) either $J_1<2048/729$, or $J_1=16/3$ and f is $P_1(x^2+xy+y^2+zt)$ or $J_1=4$ and f is $P_1(x^2+y^2+zt)$ and (b) either $J_2<7.6$ or $J_2=256/27$ and f is $\frac{1}{3}P_2(x^2+xy+y^2+3zt)$.

J. F. Koksma (Amsterdam).

Churchhouse, R. F. An extension of the Minkowski-Hlawka theorem. Proc. Cambridge Philos. Soc. 50, 220–224 (1954).

Let R be an n-dimensional region, symmetric about the origin, of content $V(R)$ and critical determinant $\Delta(R)$. Put $V(R)/\Delta(R)=Q(R)$. If R is convex, the Minkowski-Hlawka theorem [Hlawka, Math. Z. 49, 285–312 (1943); these Rev. 5, 201] asserts that $Q(R)\geqq 2\zeta(n)$. The non-convex two-dimensional case is considered in the present paper. Let g be any real-valued function such that $g(0)=1$, $g(1)=0$, g is non-increasing and concave upward for $0<x<1$, and $g(x)=g^{-1}(x)$ for $0<x<1$. Let α be the unique solution in $0<x<1$ of the equation $x=g(x)$. Then if R is the region containing the origin and bounded by the arcs $|y|=g(|x|)$, and $V(R)\geqq 2\alpha$, then $Q(R)\geqq 4$. The constant 4 is best possible, since equality holds when R is the square $|x|+|y|\leqq 1$.

W. J. LeVeque (Ann Arbor, Mich.).

Bambah, R. P. On polar reciprocal convex domains. Proc. Nat. Inst. Sci. India 20, 119–120 (1954).

Let K be a centrally symmetric convex body with center at the origin in Euclidean space E_n. A lattice is K-admissible if it contains no points of K except the origin. A lattice is K-covering if every point of E_n lies in a translate of K by an element of the lattice. The critical determinant $\Delta(K)$ is the minimal volume of the fundamental parallelotope for all K-admissible lattices, the covering constant $c(K)$ is the maximal volume of the fundamental parallelotope for all K-covering lattices. The author proves: If k, K are centrally symmetric convex domains in E_2 with centers at the origin, which are polar reciprocal with respect to the unit circle C with center at the origin, then $2\leqq\Delta(K)c(k)\leqq 9/4$. These bounds are proved best possible by considering the inscribed and circumscribed square of C for the lower bound and the inscribed and circumscribed regular hexagon of C for the upper bound. Generalizations to higher dimensions are indicated. *E. G. Straus* (Los Angeles, Calif.).

Sawyer, D. B. On the covering of lattice points by convex regions. Quart. J. Math., Oxford Ser. (2) 4, 284–292 (1953).

Let K be a closed central convex region in the euclidean plane, such that, however it is displaced in the plane, at least one lattice point is covered. Here displacements include both translations and rotations. The author proves that the area of K is at least 4/3, and that this minimum value is attained only if K is congruent to the region K^*, defined by $|y|\leqq\frac{3}{4}-x^2$, $|x|\leqq\frac{1}{2}$. The special position of K^* is connected to the fact that to every point A on one of the vertical sides there can be found points B and C on the curved sides such that AB and AC are perpendicular and both of length 1. *N. G. de Bruijn* (Amsterdam).

Goddard, L. S. Approximation to π by trigonometrical surds. Quart. J. Math., Oxford Ser. (2) 4, 308–313 (1953).

The author gives two systematic methods for obtaining algebraic numbers approximating π. The first method is to approximate θ by a trigonometric polynomial $\theta\sim\sum_{r=1}^{s}a_r \sin r\theta$, where a_r are rational, and then substitute for θ a rational multiple of 2π. The terms on the right become algebraic numbers whose sum approximates a rational multiple of π.

The second method is based on a continued fraction transformation of the identity

$$\theta=\sin\theta\cos\theta\sum_{n=0}^{\infty}4^n(n!)^2(\sin^{2n}\theta)/(2n+1)!$$

so that the successive convergents, which are rational in $\sin\theta$ and $\cos\theta$, give algebraic approximations to θ whenever θ is made a rational multiple of 2π. Apparently the rules of the game permit only values of θ of the form $k\pi/60$ where k

is an integer. At any rate these cases give the more elegant results such as the approximation

$$\frac{427+101\sqrt{3}}{105+50\sqrt{3}}=3.141592642.$$

D. H. Lehmer (Berkeley, Calif.).

Oppenheim, A. On the representation of real numbers by products of rational numbers. Quart. J. Math., Oxford Ser. (2) 4, 303–307 (1953).

The author, generalizing Cantor's representation of real numbers as infinite products, proves among others the following theorems. Given integers $n_i\geq 1$ $(i=1, 2, \cdots)$, there exist for any $x>1$ integers $d_i\geq 1$ $(i=1, 2, \cdots)$ such that

$$(1)\qquad x=\prod_{i=1}^{\infty}\left(1+\frac{n_i}{d_i}\right),$$

$$(2)\qquad 1\leq d_1\leq d_2\leq\cdots,\quad d_{i+1}>\frac{n_{i+1}}{n_i}(d_i-1)(d_i+n_i).$$

The d_i are determined by the following recursion formulas:

$$(3)\qquad x_1=x,\quad d_i=1+\left[\frac{n_i}{x_i-1}\right],\quad x_{i+1}=\frac{d_ix_i}{d_i+n_i}.$$

If $n_i=c_1c_2\cdots c_i$, c_i integers and $x=\prod_{i=1}^{\infty}(1+n_i/d_i)$ and (2) holds, then the d_i are given by (3). *P. Erdös.*

ANALYSIS

Paasche, I. Über die Cauchysche Mittelwertefunktion (Potenzmittel). Math. Naturwiss. Unterricht 6, 362–364 (1954).

Im folgenden werden je 2 verschiedene Beweise für die Hebbarkeit der Unstetigkeit sowie für die Monotonie der Mittelwertefunktion $m(x)=m(x, z_\nu)=[(z_1^x+\cdots+z_n^x)/n]^{1/x}$, alle $z_\nu>0$, $-\infty<x<+\infty$ gegeben.

Extract from the paper.

Bonsall, F. F. Corrigendum: The characterization of generalized convex functions. Quart. J. Math., Oxford Ser. (2) 4, 253 (1953).

See same J. (2) 1, 100–111 (1950); these Rev. 12, 83.

Lekkerkerker, C. G. A property of logarithmic concave functions. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. 56 = Indagationes Math. 15, 505–513, 514–521 (1953).

A positive-valued function $f(x)$ is said to be logarithmic convex (logarithmic concave) in an interval if $\log f(x)$ is convex (concave) there. If $f(x, t)$ is a logarithmic convex function of x in (a, b) for each t in (c, d) and $g(x)=\int_c^d f(x, t)dt$ exists in (a, b), then Hölder's inequality shows that $g(x)$ is logarithmic convex in (a, b). However, this proposition does not hold if "logarithmic convex" is replaced by "logarithmic concave". It is proved that if $f(x)$, $\varphi(x)$ are positive, steadily decreasing and logarithmic concave for $0\leq x<\infty$, then $f_1(x)=\int_0^\infty f(t)\varphi(t+x)\,dt$ exists and has the same properties. The proof depends on approximating $f(x)$ and $\varphi(x)$ by functions $F(x)$ and $\Phi(x)$ such that the graphs of $\log F(x)$, $\log\Phi(x)$ are polygons inscribed in the graphs of $\log f(x)$ and $\log\varphi(x)$. It is further proved that if $\log\varphi(x)$ is not linear in any interval $a\leq x<\infty$, then $f_1(x)$ is strictly logarithmic concave. An application to statistics is given.

F. F. Bonsall (Newcastle-upon-Tyne).

Bloom, Melvin. On the total variation of solutions of the bounded variation moment problem. Proc. Amer. Math. Soc. 4, 118–126 (1953).

Given a sequence of reals $\{\mu_n\}$ $(n=0, 1, \cdots)$ and a number a $(-\infty\leq a<\infty)$ the problem is the existence and construction of functions $\varphi(t)$ of bounded variation such that $\mu_n=\int_a^\infty t^n d\varphi(t)$ $(n=0, 1, \cdots)$, the integrals converging absolutely; this is called the problem (μ, a). Boas [Bull. Amer. Math. Soc. 45, 399–404 (1939)] first showed that $(\mu, 0)$ always has solutions. Pólya [C. R. Acad. Sci. Paris 207, 708–711 (1938); Comment. Math. Helv. 11, 234–252 (1939)] showed that every problem $(\mu, -\infty)$ has solutions which are entire functions and also solutions which are step-functions with a preassigned set of possible points of discontinuity, this set being subject to the condition of having no finite limit point. The author shows that also the problem $(\mu, 0)$ has entire solutions as well as step-function solutions which may also be subjected to the additional restriction $\int_{-\infty}^{\infty}|d\varphi(t)|<|\mu_0|+\epsilon$, where ϵ is a preassigned positive number. The methods used are close to Pólya's with appropriate modifications so as to yield the new restriction.

I. J. Schoenberg (Philadelphia, Pa.).

Khamis, Salem H. On the reduced moment problem. Ann. Math. Statistics 25, 113–122 (1954).

Let $\Phi(x)$ be a cumulative distribution function (c.d.f.) on $(-\infty, \infty)$ and $\mu_r(\xi)$ be the corresponding rth order moment about ξ. By combining classical results for the reduced moment problem with certain theorems on persymmetric determinants the author shows that, if $\Psi(x)$ is another c.d.f. having the same moments of orders $0, 1, \cdots, 2n$ as $\Phi(x)$, and if each of $\Phi(x)$ and $\Psi(x)$ has at least $n+1$ points of increase, then (*) $|\Phi(x)-\Psi(x)|\leq\rho_n(x)$, where $\rho_n(x)=|\mu_{i+j}(\xi)|_0^n/|\mu_{i+j}(x)|_1^n$ and ξ may be any real number. He also obtains certain refinements upon (*) under additional assumptions concerning $\Phi(x)$ and $\Psi(x)$. For example, in the case where $\Phi(x)$ and $\Psi(x)$ are continuous everywhere, differentiable on a finite interval $[c, d]$, and constant outside it, and where $1-A_0=\inf\Phi'(x)/\Psi'(x)<1$ and $1-B_0=\inf\Psi'(x)/\Phi'(x)<1$ $(c\leq x\leq d, \Phi'(x)+\Psi'(x)\neq 0)$, he shows (Theorem 4) that $|\Phi(x)-\Psi(x)|\leq K\rho_n(x)$, with $0<K=A_0B_0/(A_0+B_0-A_0B_0)\leq 1$. He goes on to apply these refinements to distributions of various special types, obtaining incidentally certain crude inequalities for orthogonal polynomials (relative to a distribution on a finite interval). *H. P. Mulholland* (Birmingham).

Viola, Tullio. Limitazioni per i momenti del quart'ordine d'una funzione, definita nello spazio euclideo ad n dimensioni ed ivi limitata. Ann. Scuola Norm. Super. Pisa (3) 7, 79–89 (1953).

Let $\{m_{ij}\}$ be a given positive definite symmetric matrix, and $\{m_{ijst}\}$ a symmetric array. When does there exist a continuous function f defined in E_n, with $0\leq f\leq L$, with vanishing first order moments, and whose second and fourth order moments coincide with m_{ij} and m_{ijst}, respectively? The author derives as a necessary condition that a certain matrix depending on the m_{ij} and m_{ijst} be positive definite.

W. Feller (Princeton, N. J.).

Lalaguë, Pierre. Sur des classes de fonctions indéfiniment dérivables. C. R. Acad. Sci. Paris 238, 761–762 (1954).

The author states a refinement of a theorem announced previously [same C. R. 236, 2473–2475 (1953); these Rev.

15, 107], and states two further results on classes of infinitely differentiable functions. *S. Agmon.*

Calculus

***Abdelhay, J. Curso de análise matemática. Vol. II.** [Course of mathematical analysis. Vol. II.] 2d ed. Universidade do Brasil, Rio de Janeiro, 1953. vi+280 pp. Cr $150.00.

[For vol. I see these Rev. 14, 959.] Contents: Partial derivatives, implicit functions, line integrals and differential forms, elements of differential geometry, double integrals. *R. P. Boas, Jr.* (Evanston, Ill.).

Mikeladze, Š. E. Expansion of the finite difference of a function in differences of its derivative. Doklady Akad. Nauk SSSR (N.S.) **92**, 479–482 (1953). (Russian)

Expressions of the following form are derived explicitly.

$$\Delta^{(n)}f(a+\lambda h)=h^n\sum_{\rho=0}^{r}A_{n,\rho}\Delta^{(\rho)}f^{(n)}(a)+R_{n,r};$$

$$\delta^{(2n-1)}f(a+h/2)=h^{2n-1}\sum_{\rho=1}^{r}B_{n,\rho}\mu\delta^{(2\rho-2)}f^{(2n-1)}(a+h/2)+R_{n,r}.$$

P. Davis (Washington, D. C.).

Krafft, Maximilian. Elementare Ermittlung des Wertes des Integrals $\int_0^\infty e^{-x^2}dx$. Jber. Deutsch. Math. Verein. **57**, Abt. 1, 31–33 (1954).

Livingston, A. E. A necessary condition for the convergence of $\int_a^\infty f(x)dx$. Amer. Math. Monthly **61**, 250–251 (1954).

***Valentiner, Siegfried. Vektoranalysis.** 7. Aufl. Sammlung Göschen Bd 354. Walter de Gruyter and Co., Berlin, 1954. 138 pp. DM 2.40.

***Lyra, G. Vektor- und Tensorrechnung.** Naturforschung und Medizin in Deutschland, 1939–1946, Band 3. Angewandte Mathematik, Teil I, pp. 263–268. Verlag Chemie, Weinheim, 1953. DM 20.00.

Castoldi, Luigi. Una nuova formula integrale nell'analisi dei vettori—applicazioni fisico-matematiche ed estensione agli spazi di Riemann. Atti Accad. Ligure **8** (1951), 330–336 (1952).

The author proves the following well-known generalized theorem of Stokes ($n=3$):

$$\int_{\tau_1}u^i df_{ih}=\int_{\tau_2}[(\partial_i u^i)df_h-(\partial_h u^i)df_i],$$

where τ_1 is the boundary of the surface τ_2. An application is given in the theory of electromagnetism. The theorem can be generalized for euclidean spaces but not for Riemannian spaces. *J. Haantjes* (Leiden).

Klamkin, Murray S. On vector sums and products. Amer. J. Phys. **22**, 159–161 (1954).

Theory of Sets, Theory of Functions of Real Variables

Froda, Alexandru. Sur les ensembles extraits des familles d'ensembles. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 701–711 (1952). (Romanian. Russian and French summaries)

Soient Z et $F=\{E_\zeta|\zeta\in Z, E_\zeta\neq E_{\zeta'}$ si $(\zeta,\zeta')_{\neq}\subseteq Z\}$ un ensemble non vide et une famille d'ensembles non vides, respectivement. Un ensemble X contenant un seul point x_ζ de chaque E_ζ s'appelle un extrait de F; l'extrait X est propre si $x_\zeta\neq x_{\zeta'}$ pour $\zeta\neq\zeta'$; si X n'est pas propre, il est dit impropre. F appartient à la classe (P), s'il existe un extrait propre de F; dans le cas contraire, l'A. dit que F appartient à la classe (I), symboliquement $F\in$ (I). Pour que $F\in$ (I), il faut et il suffit qu'il existe une partie non vide A de F et un $E\in F-A$ tels que E soit inclus dans chaque extrait de A (Th. 4.1). Si $F\in$ (I), il y a $0\subset A\subset F$ tel que chaque $B\in A$ soit contenu dans la réunion des éléments de A distincts de B (Th. 3.II). Si Z est un segment initial transfini de nombres ordinaux et si $F\in$ (I), il y a un $\zeta\in Z$ tel que $\overline{\overline{E}}_\zeta\leq\zeta$ (Th. 3.1). L'existence d'un extrait pour chaque famille est équivalente à l'axiome de choix. *G. Kurepa* (Zagreb).

Barbălat, I. Les théorèmes de Zorn et Kneser dans la théorie des ensembles ordonnés. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 751–762 (1952). (Romanian. Russian and French summaries)

The paper is connected with two objections to reasoning by Witt [Math. Nachr. **4**, 434–438 (1951); these Rev. **12**, 596] concerning Zorn's Lemma and Kneser's Lemma [if each well-ordered subset of an ordered set S is bounded from above, then each mapping f of S into itself such that $fx\geq x$ has at least one fixed point; cf. Math. Z. **53**, 110–113 (1950); these Rev. **12**, 323]. The proof of Kneser's Lemma is based upon this statement: If S is inductive and contains the upper boundary of each bounded subset, and if for each nonvoid chain $L\subseteq S$ the relation $x<\sup L$ implies the existence of an $l_0\in L$ so that $x<l_0$, then there is no mapping f from S to S so that $fx>x$ and so that $x<f(y)$ implies $x\leq y$. The content and reasoning contained in the paper are quite similar to those in Inagaki, Math. J. Okayama Univ. **1**, 167–176 (1952) [these Rev. **14**, 27]. *G. Kurepa.*

Sierpiński, W. Une généralisation des théorèmes de S. Mazurkiewicz et F. Bagemihl. Fund. Math. **40**, 1–2 (1953).

The author proves the following theorem: Associate with each line l of the plane a cardinal number m_l satisfying $2\leq m_l\leq 2^{\aleph_0}$. Then there exists a set S in the plane so that for each l the cardinal number of the intersection of S with l equals m_l. *P. Erdös* (South Bend, Ind.).

Brods'kiĭ, M. S. On harmonic functions on Baire spaces. Dopovidi Akad. Nauk Ukrain. RSR **1952**, 377–380 (1952). (Ukrainian. Russian summary)

Let n_k be a sequence of integers and let Q be the space of sequences x of integers satisfying $0<x_k<n_k$ for all k, metrized by $d(x,y)=1/k$ if k is the smallest integer for which $x_k\neq y_k$. Then Q, called a Baire space in this paper, is compact and there is a measure on Q invariant under all isometries of Q. It is shown here that for n_k a bounded sequence every continuous function on Q can be represented by a uniformly convergent "Fourier" series in terms of a natural orthogonal system; this generalizes the case $n_k=2$ for all k, which can be reduced to the Haar orthogonal system. *M. M. Day* (Urbana, Ill.).

Marczewski, E. On compact measures. Fund. Math. **40**, 113–124 (1953).

A class F of subsets of a set X is called compact if every countable subclass of F that has the finite intersection property has a non-empty intersection. A measure is a non-negative, additive, and normalized set function on a Boolean algebra M of subsets of X. A class F approximates M with respect to a measure μ if, for every A in M and every positive number ϵ, there exist sets B and C, in M and F respectively, such that $B\subset C\subset A$ and $\mu(A-B)<\epsilon$. A measure μ on M is compact if there exists a compact class F which approximates M with respect to μ.

The purpose of the paper is to develop the basic properties of compact measures and to apply them to the theory of measures in product spaces. The key fact about compact measures is that they are countably additive. From this it is deduced that every compact measure has a compact (and therefore countably additive) extension to the Boolean σ-algebra generated by its domain.

If $\{X_t\}$ is a family of sets, a subset Z of their Cartesian product is called a countably reduced Cartesian product if Z contains points with arbitrarily prescribed coordinates at any countable set of indices. For each subset A of some X_t the set $C_t(A)$ is defined to be the set of all those points x in Z for which $x_t\,\epsilon\, A$. If, for each t, M_t is a Boolean algebra of subsets of X_t and μ_t is a measure on M_t, then M_t^* denotes the class of all sets of the form $C_t(A)$ with A in M_t, and μ_t^* is the measure on M_t^* defined by $\mu_t^*(C_t(A))=\mu_t(A)$. A product of the measures u_t is defined to be a common extension of the measures u_t^* to the Boolean algebra generated by all the M_t^*.

The first main result is that every product of compact measures is compact. The second main result is a generalization of Kolmogoroff's extension theorem and can be stated as follows. For every finite subset T_0 of the set T of indices let $L(T_0)$ be the Boolean σ-algebra generated by all the M_t^* with t in T_0 and let L be the union of all the $L(T_0)$. If μ is a measure on L such that $\mu|L(T_0)$ is always countably additive and $\mu|M_t^*$ is always compact, then μ itself is compact. *P. R. Halmos* (Chicago, Ill.).

Marczewski, E., and Ryll-Nardzewski, C. Projections in abstract sets. Fund. Math. **40**, 160–164 (1953).

Suppose that X and Y are sets, E and F are classes of subsets of X and Y respectively, and H is the class of all Cartesian products $A\times B$ with A in E and B in F. The operation of projection, from $X\times Y$ to X, does not preserve Boolean operations, and, consequently classes obtained from H by such operations do not necessarily project onto classes similarly obtained from E. With measure-theoretic applications in mind (see following review), the authors obtain some positive results in this direction in case the class F is compact (see preceding review). Sample: if F is compact, then the projection into X of every non-empty set that is a countable intersection of finite unions of sets in H is a countable intersection of finite unions of sets in E.
P. R. Halmos (Chicago, Ill.).

Marczewski, E., and Ryll-Nardzewski, C. Remarks on the compactness and non direct products of measures. Fund. Math. **40**, 165–170 (1953).

If X and Y are sets and if μ and ν are measures in X and Y respectively, a measure λ in $X\times Y$ is called a product of μ and ν if, identically, $\lambda(A\times Y)=\mu(A)$ and $\lambda(X\times B)=\nu(B)$. The authors prove that if μ is countably additive and ν is compact (for terminology see the second preceding review), then λ is countably additive. If, moreover, the domains of μ and ν are Boolean σ-algebras and if the domain of λ is the Boolean σ-algebra generated by rectangles with sides in the domains of μ and ν, then, identically, $\lambda_*(A\times Y)=\mu_*(A)$, where the asterisk denotes inner measure. In neither statement is it possible to replace compactness by countable additivity. *P. R. Halmos* (Chicago, Ill.).

Ryll-Nardzewski, C. On quasi-compact measures. Fund. Math. **40**, 125–130 (1953).

The only measures considered in this paper are countably additive measures on Boolean σ-algebras; otherwise the terminology agrees with that in the preceding three reviews. A measure μ on M is called quasi-compact if, corresponding to any countable subclass F of M and to any positive number ϵ, there exists a set A_0 in M such that $\mu(A_0)>1-\epsilon$ and such that the sets $A_0\cap A$, with A in F, form a compact class. The following statements constitute a fair sample of the author's results. The measure μ is quasi-compact if and only if for each real-valued measurable function f there exists a set A in M such that $\mu(A)=1$ and such that $f(A)$ is a Borel set. Another necessary and sufficient condition for the quasi-compactness of μ is that the restriction of μ to every countably generated σ-subalgebra of M be compact.

The author asserts that, in the presence of a suitable separability assumption, compactness, quasi-compactness, and (almost) isomorphism with Lebesgue measure are equivalent concepts. To the reviewer it appears that non-atomicity should be added to the assumptions.
P. R. Halmos (Chicago, Ill.).

Nedoma, Jiří. Convergence of sequences of measures. Čehoslovack. Mat. Ž. **2**(77), 239–242 (1952). (Russian. English summary)

Let A be a σ-field of sets. It is shown that there is a sequence μ_n of measures on A such that $\mu_n(e)$ tends to $\mu(e)$ for each e in A, but the convergence is not uniform if and only if there is a non-atomic measure on A.
M. M. Day (Urbana, Ill.).

Krickeberg, Klaus. La nécessité de certaines hypothèses de Vitali fortes dans la théorie de la dérivation extrème de fonctions d'intervalle. C. R. Acad. Sci. Paris **238**, 764–766 (1954).

Let p denote the proposition: The integral of every Lipschitzian function of intervals is not less than that of its upper derivative. Subject to a very general definition of interval and to the restriction that the measure be finite, the author's main statement is that p implies Vitali's theorem but only in a weak form. *L. C. Young.*

Morse, Anthony P. Dini derivatives of continuous functions. Proc. Amer. Math. Soc. **5**, 126–130 (1954).

The following theorem is proved: If $f(x)$ is continuous, if the set $E_x(D^+f(x)\geqq\lambda)$ is dense and the set $E_x(D^+f(x)<\lambda)$ (which is of the first category) is not empty, then the set $E_x(D^+f(x)=\lambda)$ has the power of the continuum. The author also remarks on some consequences of his theorem: If $f(x)$ is a continuous non-differentiable function, then $E_x(D^+f(x)<+\infty)$ is a first category set with the power of the continuum; if $f(x)$ is continuous and the set $E_x(D^+f(x)<0)$ has power less than the continuum, then $f(x)$ is non-decreasing. He points out that the example of Cantor's function shows that his theorem is false when D^+ is replaced by D_+. *U. S. Haslam-Jones* (Oxford).

Racine, C. A note on the theory of the Riemann integral. Math. Student **21** (1953), 97–103 (1954).

The note gives simple proofs of a number of theorems on Riemann integration. There is a simple demonstration of the fundamental existence theorem of the integral (in which the condition that $f(x)$ be bounded is omitted) as well as a refinement of a theorem of Jordan [Cours d'analyse, t. II, 3rd ed., Gauthier-Villars, Paris, 1913, p. 189] which reads: Let $f(x, y)$ be integrable in y on $c \leqq y \leqq d$, for every x on $a \leqq x \leqq b$. Then the function $F(x)=\int_c^d f(x, y)dy$ is continuous in x at x_0 if $f(x, y)$ is continuous in (x, y) at (x_0, y) for all $c \leqq y \leqq d$ except at most a set of zero measure in y. To make the proof valid, the condition that $f(x, y)$ be bounded in (x, y) for $x_0-\delta \leqq x \leqq x_0+\delta$, $c \leqq y \leqq d$ for some $\delta>0$ must be added. As a consequence $\int_a^b f(x+t)dx$, $\int_a^b |f(x+t)-f(x)|dx$ and $\int_a^b f(x)f(x+t)dx$ are continuous in t. The fact that $\lim_{t\to 0}\int_a^b |f(x+t)-f(x)|dx=0$ is easily proved directly.

T. H. Hildebrandt (Ann Arbor, Mich.).

James, R. D. Generalized *n*th primitives. Trans. Amer. Math. Soc. **76**, 149–176 (1954).

The present paper is a continuation of two previous ones [James and Gage, Trans. Roy. Soc. Canada. Sect. III. (3) **40**, 25–35 (1946); James, Canadian J. Math. **2**, 297–306 (1950); these Rev. **9**, 19; **12**, 94]. In the latter paper a process of integration of order two is investigated which is called P^2 and which may be roughly described as an operation inverse to the Schwarz second derivative $D_2F(x)=\lim_{h\to 0} h^{-2}\{F(x+h)+F(x-h)-2F(x)\}$. The whole problem has its origin in the theory of representation of functions by trigonometric series and had been initiated by Denjoy; Denjoy's fundamental results will be found in his recent book "Leçons sur le calcul des coefficients d'une série trigonométrique" [4ième partie, Gauthier-Villars, Paris, 1949; these Rev. **11**, 99]. James's approach is different from Denjoy's and uses the idea (of de la Vallée Poussin and Perron) of major and minor functions. In the present paper the author gives a theory of integration P^n, $n\geqq 2$, an operation inverse to taking the nth generalized derivative of Peano-de la Vallée Poussin. According to the original definition of Peano, the number $\alpha_n=D_nF(x)$ is called the nth derivative of F at the point x if

$$F(x+t)=\alpha_0+\alpha_1 t+\cdots+\alpha_n t^n/n!+o(|t|^n)$$

for $|t|$ small, where $\alpha_0, \alpha_1, \cdots, \alpha_n$ are constants. The author shows that his integration P^n based on this definition of derivative is essentially equivalent to a definition previously introduced by Burkill [Proc. London Math. Soc. (2) **39**, 541–552 (1935)]. The definition of Peano has, however, little application to trigonometric series (though of considerable importance to power series in $e^{i\alpha}$) which require symmetric derivatives. The modification was introduced by de la Vallée Poussin who defined derivatives of even and odd orders by different formulas; the derivative $\alpha_{2m}=D_{2m}F(x)$ of order $2m$ is defined by the equation

$$\tfrac{1}{2}\{F(x+t)+F(x-t)\}=\alpha_0+\frac{\alpha_2}{2!}t^2+\cdots+\frac{\alpha_{2m}}{(2m)!}t^{2m}+o(t^{2m}),$$

and for odd order we have a similar definition with $\frac{1}{2}\{F(x+h)-F(x-h)\}$ on the left. The author correspondingly has different definitions for integrations P^{2m} and P^{2m+1} but it is shown that the two scales of integration fit together. The actual definition of major and minor function and of the generalized nth primitive are too long to be given here. The author promises applications to trigonometric series.

A. Zygmund (Cambridge, England).

Guglielmino, Francesco. Studio di una espressione legata ad una trasformazione di coppie di funzioni di due variabili. Boll. Accad. Gioenia Sci. Nat. Catania (4) **2**, 221–231 (1953).

Schwartz, J. The formula for change in variables in a multiple integral. Amer. Math. Monthly **61**, 81–85 (1954).

Theorem. Let D_1 and D_2 be open sets in Euclidean n-space E^n, and let $h: D_1\to D_2$ be a 1-1 mapping of D_1 onto D_2 such that h and its inverse h^{-1} are continuous and have continuous derivatives. Let $J(x)=|\det j(x)|$ where $j(x)$ denotes the Jacobian matrix of h. Then a function $f(x)$ is integrable over the domain D_2 if and only if the function $f(h(x))J(x)$ is integrable over D_1, and we have $\int_{D_2} f(x)dx=\int_{D_1} f(h(x))J(x)dx$. Sketch of the proof. f is assumed to be $\geqq 0$. The theorem is first proved for linear maps by reducing them to basic types. C denoting a cube in D_1 with center x, the Lebesgue measure $\mu(h(C))\leqq \max_{y\in C} N(j(y))\mu(C)$, ($\mathfrak{I}$), for a suitable matrix norm N. Then C is subdivided into non-overlapping cubes C_i with centers x_i and ($\mathfrak{I}$) applied to each C_i mapped by $j^{-1}(x_i)$. Addition and passage to the limit yield $\mu(h(C))\leqq\int_C J(x)dx$, hence the integral inequality $\int_{D_2} f(x)dx\leqq\int_{D_1} f(h(x))J(x)dx$, which is then applied to the inverse mapping h^{-1}. Remark by the reviewer: O. Haupt's interior parallelogram technique [Differential- und Integralrechnung, 3 vols., de Gruyter, Berlin, 1938], avoided by the author, leads to a stronger form of the theorem, assuming h to be a locally Lipschitzian 1-1 mapping of D_1 onto D_2.

Chr. Pauc (Nantes).

Išlins'kiĭ, O. Yu. On the transformation of a double curvilinear into a double surface integral. Dopovidi Akad. Nauk Ukrain. RSR **1951**, 397–400 (1951). (Ukrainian. Russian summary)

Elementary transformation, based on Stokes' theorem, of the repeated curvilinear integral of $ds\,ds'/R$ where R is the distance in 3-space.

L. C. Young (Madison, Wis.).

Krickeberg, Klaus. Über den Gaussschen und den Stokesschen Integralsatz. I. Math. Nachr. **10**, 261–314 (1953).

In this paper a systematic investigation is made of the Gauss formula (*) $\int_G \partial_1\alpha dx=\int_F \alpha\sigma d\mu$ where x is n-dimensional Lebesgue measure, μ is a measure defined over the Borel subsets of F, G is an open subset of real n-dimensional Euclidean space and F is its frontier, σ is a measurable function defined over F, and $\partial_1\alpha$ is a partial derivative in a fixed direction of a function α defined over G. The author considers three classes of conditions: (a) those on G, (b) those which connect μ and σ, (c) those on α. He shows that these conditions are necessary for the truth of (*) as well as sufficient and that there are various alternative forms whose logical connections with (a), (b), (c) are also given. The conditions are complicated and it is not possible to give the details in a short review. The author also studies the formula (**) $\int_G \operatorname{div} a\,dx=\int_F \langle a, s\rangle d\mu$, where s is a unit vector in the direction of the outer normal to G and $\langle a, s\rangle$ denotes the scalar product, and using his results obtained for (*) reaches analogous conclusions about (**). Reference is made to some but not to all the previous relevant work.

H. G. Eggleston (Cambridge, England).

Cesari, Lamberto. Contours of a Fréchet surface. Rivista Mat. Univ. Parma **4**, 173–194 (1953).

Let S be a Fréchet surface of the type of the 2-cell in Euclidean 3-space, and denote its point set by $[S]$. Assume

that $f(p)$, $p \varepsilon [S]$, is a real single-valued continuous function, and let t', t'' be the minimum, maximum values which f assumes on $[S]$. Consider any representation for S, T: $p=T(w)$, $w \varepsilon J$, where J is a simple closed Jordan region. For each value of t let D_t^- denote the set of points w on J where $f[T(w)]$ is less than t, let $\{\alpha\}_t$ be the collection of components α of D_t^-, and let $\{\gamma\}_\alpha$ represent the collection of components γ of the boundary of α in J. For each α in $\{\alpha\}_t$ and each γ in $\{\gamma\}_\alpha$ a quantity $\lambda(\gamma, \alpha)$ is defined in terms of the images under T of points in γ which correspond to ends in a certain set $A(\gamma, \alpha)$ which is connected and open in J and has γ as its sole boundary in J, in a manner analogous to that of Jordan length. Then

$$l(t)=l(t; T, J)=\sum_{\alpha \varepsilon \{\alpha\}_t} \sum_{\gamma \varepsilon \{\gamma\}_\alpha} \lambda(\gamma, \alpha)$$

is termed the generalized length of the image under T of the boundary of D_t^- in J. If T': $p=T'(w')$, $w' \varepsilon J'$, is a second representation for S then there is a natural correspondence between the sets $\{\alpha\}_t$, $\{\gamma\}_\alpha$ relative to T and the sets $\{\alpha'\}_t$, $\{\gamma'\}_{\alpha'}$ relative to T'. For each t satisfying $t'<t<t''$ it is shown that $\lambda(\gamma, \alpha)=\lambda(\gamma', \alpha')$ for corresponding elements. Thus it follows that for each t satisfying $t'<t<t''$ the generalized length $l(t)$ is independent of the representation for S.
P. V. Reichelderfer (Columbus, Ohio).

Fullerton, R. E. On the rectification of contours of a Fréchet surface. Rivista Mat. Univ. Parma 4, 207–212 (1953).

The assumptions, definitions, and notations are as in the preceding review. Fix a representation for S, T: $p=T(w)$, $w \varepsilon Q$, where Q is the unit square in the plane. Let ϵ be a positive real number, and consider a countable sequence of reals t_i satisfying $t'<t_i<t''$ and $l(t_i)<\infty$. Then there exists a representation for S, T_ϵ: $p=T_\epsilon(w)$, $w \varepsilon Q$, such that $|T(w)-T_\epsilon(w)|<\epsilon$, $w \varepsilon Q$, and each component of the boundary in Q of the set of points w on Q where $f[T_\epsilon(w)]$ is less than t_i is either a simple arc, a simple closed curve, or a continuum of constancy for T_ϵ on Q.
P. V. Reichelderfer (Columbus, Ohio).

Theory of Functions of Complex Variables

***Carathéodory, C. Theory of functions of a complex variable. Vol. 1.** Translated by F. Steinhardt. Chelsea Publishing Co., New York, N. Y., 1954. xiii+301 pp.

Translation of the author's Funktionentheorie, Bd I [Birkhäuser, Basel, 1950; these Rev. 12, 248].

Mambriani, Antonio. Sul concetto di "modulo parziale". Rivista Mat. Univ. Parma 4, 227–232 (1953).

The partial modulus of z with respect to θ, denoted by $|z|_\theta$, is defined as the orthogonal projection of the vector z on the ray $\arg w=\theta$, $|z|_\theta=|z|\cos(\theta-\arg z)$. The author shows the importance of this concept and how it can be used in discussing the geometry of the complex plane.
E. Hille (New Haven, Conn.).

Taĭmanov, A. D. On a problem of N. N. Luzin. Uspehi Matem. Nauk (N.S.) 8, no. 5(57), 169–171 (1953). (Russian)

In answer to a question raised by N. Luzin [Uspehi Matem. Nauk (N.S.) 7, no. 2(48), 7–16 (1952); these Rev. 13, 810] the author constructs a function $f(z)$ continuous in a domain D such that the set of monogenicity of $f(z)$ at a point z in D contains a circle. From this it is possible to construct continuous functions whose sets of monogenicity have certain other desired properties. *A. J. Lohwater.*

Kudryavcev, L. D. On harmonic mappings. Doklady Akad. Nauk SSSR (N.S.) 92, 469–471 (1953). (Russian)

Let $f(\theta)=u(\theta)+iv(\theta)$ be a homeomorphism of $|z|=1$ onto a simple closed curve bounding a region G; it is assumed that $f'(\theta)$ satisfies a Hölder condition. In order that there exist a one-to-one harmonic mapping $f(z)=u(z)+iv(z)$ ($u(z)$ and $v(z)$ harmonic in $|z|<1$) of $|z|<1$ onto G, continuous on $|z|\leq 1$, coinciding with $f(\theta)$ on $|z|=1$, and such that $\min_{|z|\leq 1}|\partial(u, v)/\partial(x, y)|>0$, it is asserted that it is necessary and sufficient that

$$\frac{d\bar{u}(\theta)}{d\theta}\frac{dv(\theta)}{d\theta}-\frac{du(\theta)}{d\theta}\frac{d\bar{v}(\theta)}{d\theta}\neq 0 \quad (0\leq\theta\leq 2\pi).$$

(Here $\bar{u}(\theta)$ denotes the function whose Fourier series is conjugate to that of $u(\theta)$.) Other properties of harmonic mappings are announced most of which are properties of interior mappings. *A. J. Lohwater* (Ann Arbor, Mich.).

Tanaka, Chuji. Note on Dirichlet series. VI. On the integral functions defined by Dirichlet series. II. Mem. Fac. Sci. Eng. Waseda Univ. 17, 85–94 (1953).

This is a continuation of Note V of the same series [Tôhoku Math. J. (2) 5, 67–78 (1953); these Rev. 15, 206]. It is supposed that the Dirichlet series $\sum_1^\infty a_n e^{-\lambda_n s}=F(s)$ converges uniformly to an entire function whose order ρ and type λ is to be determined. It should be noted that the author's terminology differs from the usual one; thus he would say that $\exp[\lambda e^{-\rho s}]$ is of order ρ and type λ. He defines

$$\rho=\limsup_{\sigma\to-\infty}(-\sigma)^{-1}\log\log^+ M(\sigma),$$

$$M(\sigma)=\sup|F(\sigma+it)|,$$

$$\lambda=\limsup_{\sigma\to-\infty} e^{\rho\sigma}\log^+ M(\sigma),$$

and proves that

$$-\rho^{-1}=\limsup_{\omega\to\infty}(\omega\log\omega)^{-1}\log T_\omega{}^k$$

$$\rho^{-1}\log(e\rho\lambda)=\limsup_{\omega\to\infty}[\omega^{-1}\log T_\omega{}^k+\rho^{-1}\log\omega]$$

where k, $0<k<1$, is fixed but arbitrary, and

$$T_\omega{}^k=\sup_t\left|\sum_{\lambda_n<\omega} a_n(\omega-\lambda_n)^k\exp(\lambda_n{}^2-\omega^2-i\lambda_n t)\right|.$$

E. Hille (New Haven, Conn.).

San Juan, Ricardo. Some theorems on differentiation of asymptotic power series. Collectanea Math. 5, 269–284 (1952). (Spanish)

Let $f(z)$ have the asymptotic power series $\sum_0^\infty a_k z^k$ with bounds m_n, so that $|f(z)-\sum_{k=0}^n a_k z^k|\cdot|z|^{-n}\leq m_n$ in a certain region. The author obtains estimates for m_n', the corresponding bounds for $f'(z)$ and the partial sums of the differentiated series, in a suitably chosen smaller region. The results take a more elegant form, for power series in $1/z$, in a region determined by $\Re(z^{1/\alpha})>a>0$ rather than in the conventional angular region. *R. P. Boas, Jr.*

Mergelyan, S. N. Uniform approximations to functions of a complex variable. Amer. Math. Soc. Translation no. 101, 99 pp. (1954).

Translated from Uspehi Matem. Nauk (N.S.) 7, no. 2(48), 31–122 (1952); these Rev. 14, 547.

Bonsall, F. F., and Marden, Morris. Critical points of rational functions with self-inversive polynomial factors. Proc. Amer. Math. Soc. 5, 111–114 (1954).

$\mathfrak{N}(f, D)$ désignant le nombre total de zéros de f dans D et $\mathfrak{Q}(f, D)$ le nombre de pôles distincts de f dans D, on a:

$$\mathfrak{N}(\phi', |z|>1)=\mathfrak{N}(\phi, |z|>1)+\mathfrak{Q}(\phi, |z|\geqq 1)$$

où $\phi=fg/FG$, f et F étant des polynomes "self-inversive" (à zéros symétriques par rapport à $|z|=1$), g et G des polynomes satisfaisant à:

$$\mathfrak{N}(g, |z|>1)=\mathfrak{N}(G, |z|<1)=0, \quad \deg fg > \deg FG.$$

Cette égalité réalise la synthèse de plusieurs résultats antérieurs [cf. J. L. Walsh, The location of critical points . . . , Amer. Math. Soc. Colloq. Publ., v. 34, New York, 1950; ces Rev. 12, 249]. *J. Lelong* (Lille).

Špak, G. S. On some estimates for the argument of an analytic function. Doklady Akad. Nauk SSSR (N.S.) 92, 711–713 (1953). (Russian)

Let N denote the class of functions $f(z)=b_0+b_1z+\cdots$, regular and zero-free in $|z|<1$, with given b_0 and b_1. Let $\alpha=\arg b_0$, $-\pi<\alpha\leqq\pi$, and let $\arg f(z)$ be defined by continuity from this. The author proves that $\sup|\arg f(z)|\geqq\rho_0$, where ρ_0 is the largest root of

$$\tfrac{1}{4}\pi\rho_0^{-1}|b_1/b_0|=\cos(\tfrac{1}{2}\pi\alpha/\rho_0),$$

with equality possible for an explicitly given $f(z)$. Proof: let ρ (assumed finite) be the maximum of $|\arg f(z)|$ ($f(z)$ not constant). Then $\phi(z)=\{f(z)/|b_0|\}^{\pi/2\rho}$ has $|\arg\phi(z)|<\pi/2$ and can be written in the form $\{1+\omega(z)\}/\{1-\omega(z)\}$, with $|\omega(z)|<1$. Known inequalities for the coefficients of bounded functions then lead to the result. In particular, $\sup|\arg f(z)|\geqq\frac{1}{4}\pi|b_1/b_0|$. It follows further that

$$\sup\arg f(z)-\inf\arg f(z)\geqq\tfrac{1}{2}\pi|b_1/b_0|,$$

with equality possible. *R. P. Boas, Jr.* (Evanston, Ill.).

Sunyer Balaguer, F. On the moments of functions holomorphic and bounded in an angle. Revista Mat. Hisp.-Amer. (4) 13, 241–246 (1953). (Spanish)

The author proves the following theorem. Let $f(z)$ be analytic for $|\arg z|<\alpha\pi/2$, and uniformly bounded in each interior angle. If

$$(*)\qquad \int_0^\infty |f(x)|x^n dx<\Gamma(\alpha' n+1), \quad 0<\alpha'<\alpha,$$

for an infinity of n, then $f(z)\equiv 0$. This generalizes a result obtained independently by San Juan [C. R. Acad. Sci. Paris 236, 1941–1943 (1953); these Rev. 14, 859]. The author's proof uses (*) to show that $f(x)$ is small on an appreciable part of the real axis, maps the angle into a circle, and then applies a theorem of Milloux [Mathematica, Cluj 4, 182–185 (1930)]. *R. P. Boas, Jr.*

Tsuji, Masatsugu. A remark on Rengel's theorem concerning Szegö's conjecture. Kōdai Math. Sem. Rep. 1953, 117–118 (1953).

Let $w=z+\cdots$ be regular and univalent in $|z|<1$, and let Γ be the boundary of the image domain. Draw n equiangular half-lines through $w=0$ and let $w_1, w_2, \cdots, w_n$ be the first points of intersection of these lines with Γ. If $d=\max(|w_j|, j=1, 2, \cdots, n)$, then $d\geqq(\frac{1}{4})^{1/n}$. The proof first given by Rengel is now simplified.
A. W. Goodman (Lexington, Ky.).

Garabedian, P. R., and Royden, H. L. The one-quarter theorem for mean univalent functions. Ann. of Math. (2) 59, 316–324 (1954).

Let $f(z)=z+a_2z^2+\cdots$ be regular in $|z|<1$ and let $n(w)$ be the number of roots of the equation $f(z)=w$ there. Then $f(z)$ is said to be mean univalent if

$$\int_0^R \rho d\rho\int_0^{2\pi} n(\rho e^{i\theta})d\theta\leqq\pi\rho^2, \quad \rho>0.$$

Let d be the smallest number such that with this hypothesis $n(w)>0$, and so $n(w)=1$ for $|w|<d$. Then the authors prove that $d\geqq\frac{1}{4}$, with equality for $f(z)=z/(1-z)^2$. The result was conjectured by Spencer [Ann. of Math. (2) 42, 614–633 (1941); these Rev. 3, 78], who proved $d>1/7$. The authors first use circular symmetrisation to show that it is sufficient to suppose that the Riemann surface of $f(z)$ consists of a circle $|w|<d$, together with a portion $|\arg w|<p(R)$, $d<|w|=R<\infty$, where

$$\int_d^R (p(\rho)-2\pi)\frac{d\rho}{\rho}\leqq 0, \quad d<R<\infty.$$

They next use variational methods and polygonal approximations to discuss the case when $p(\rho)-2\pi$ has at most a fixed number m of sign changes. An ingenious use of the argument principle then allows them to deduce from their results that $p(\rho)\equiv 2\pi$ for the extremal functions in this case and so in general, and the result follows.
W. K. Hayman (Exeter).

Robertson, M. S. Multivalently star-like functions. Duke Math. J. 20, 539–549 (1953).

Let $S(p)$ denote the class of functions $f(z)=a_1z+a_2z^2+\cdots$ regular and multivalently-starlike of order p ($\geqq 1$) in $|z|<1$. Goodman and Robertson [Trans. Amer. Math. Soc. 70, 127–136 (1951); these Rev. 12, 691] proved the sharp inequality

$$|a_n|\leqq\sum_{k=1}^{p}\frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)}|a_k| \quad (n>p),$$

for the case of real coefficients a_k. This inequality is now established in the complex case, but under the restriction $a_1=a_2=\cdots=a_{p-2}=0$. The general case is still undecided.
W. W. Rogosinski (Newcastle-upon-Tyne).

Ronkin, L. I. On approximation of entire functions by trigonometric polynomials. Doklady Akad. Nauk SSSR (N.S.) 92, 887–890 (1953). (Russian)

According to Levitan [C. R. (Doklady) Acad. Sci. URSS (N.S.) 15, 169–172 (1937)], an entire function F of exponential type σ which is bounded on the real axis can be approximated on any finite interval by trigonometric sums S such that $\sup|S|\leqq\sup|F|$ and the exponents in S do not exceed σ in absolute value. Several proofs and generalizations have been given. The author outlines his work on extensions to functions of several variables. His central result will be quoted here in its one-variable form. Let $\alpha(x)$ be a positive function which is infinite at each end of the real axis; let (1) $\int_{-\infty}^{\infty}(1+t^2)^{-1}\log\alpha(t)dt<\infty$, and let $\alpha(t)$ satisfy either (2) $\alpha(t)$ is an increasing function of $|t|$ or (3) $\alpha(s)\alpha(t)\geqq\alpha(s+t)$. Then for any entire function $F(z)$ of exponential type σ such that $\sup|F(x)/\alpha(x)|=M<\infty$, there exist trigonometric sums S_h, converging uniformly to $F(x)$ on every compact set as $h\to 0$, with exponents not exceeding $\sigma+h$ in absolute value, and $\sup|S_h(x)/\alpha(x)|\leqq M$. (For functions of n variables, it is supposed that there are func-

tions α_k, satisfying (1), (2), (3), such that

$$\alpha_1(x_1)\cdots\alpha_n(x_n) \geqq \alpha(x_1, \cdots, x_n).)$$

The proof of the main theorem is stated to depend on two theorems, one involving (1) and (2), the other involving (2), (3), the first reading as follows. Let $\alpha(t)$ be majorized by $\alpha^*(t)$ satisfying (1) and (2). Then there is an entire function $\Phi(z)$ of exponential type σ such that for every real λ, $\lim \Phi(\lambda x)\alpha(x) = 0$. The proof of the main theorem is then said to be analogous to that of a similar one-variable theorem in inaccessible papers by V. A. Marčenko and by Ahiezer and Marčenko. *R. P. Boas, Jr.*

Noshiro, Kiyoshi. Theory of cluster sets of analytic functions. Sûgaku 5, 65–72 (1953). (Japanese)

An expository paper containing a survey of recent developments on cluster sets of analytic functions, with special reference to interrelations between the results established by several writers and by the author. *Y. Komatu.*

Myrberg, P. J. Über die Iteration von algebraischen Funktionen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 164, 9 pp. (1954).

This paper is concerned with the problem of studying the powers of the relation R consisting of the set of ordered pairs of complex numbers (x, y) satisfying $P(x, y) = 0$ where P is a given irreducible polynomial with complex coefficients. In particular, given x one is concerned with the set $M(x)$ of the cluster points of the set of points w which satisfy xR^nw for some integer n. The problem is treated for the special case where uniformization linearizes the problem, i.e. when $P(x, y) = 0$ admits a uniformizing pair of automorphic functions $x = f(z)$, $y = f(S(z))$ where S is a linear fractional transformation. It is shown that under this hypothesis, $M(x)$ is the extended plane except under certain special circumstances. *M. Heins* (Providence, R. I.).

Belinskiĭ, P. P. On metric properties of a quasi-conformal mapping. Doklady Akad. Nauk SSSR (N.S.) 93, 589–590 (1953). (Russian)

The author gives two counter-examples (the second of which is invalid) to a theorem of A. Pfluger [C. R. Acad. Sci. Paris 226, 623–625 (1948); these Rev. 9, 421] which asserts that, under the one-to-one continuous extension to $|z| = 1$ of a quasi-conformal one-to-one mapping of $|z| < 1$ onto $|w| < 1$, a set on $|z| = 1$ of inner measure zero is transformed into a set on $|w| = 1$ with the same property. The first counter-example shows that the definition of quasi-conformality, which Pfluger uses, does not even permit an extension to $|z| = 1$ of the mapping function. The author states three additional conditions (e.g., uniformly bounded dilatation coefficient) under which such an extension is possible and for which the conclusion of Pfluger's theorem holds. The author points out an unjustified assumption in the first step of Pfluger's proof and asserts that, under any one of the three additional hypotheses, a step-by-step recapitulation of Pfluger's proof yields the desired result. This is unfortunate, because the author seems unaware of other errors in the proof (e.g., p. 625, line 5, where the assumption is used that the linear measure of an open set is equal to the measure of its closure). *A. J. Lohwater.*

Huckemann, F. An extension of the Ahlfors distortion theorem. Proc. Cambridge Philos. Soc. 50, 261–265 (1954).

Let D denote a simply-connected region properly contained in the finite plane and let p_1, p_2 denote two distinct boundary elements. The boundary elements p_1, p_2 divide the remaining boundary elements into two classes such that no two boundary elements of the same class are separated by p_1 or p_2. These classes are termed banks. A set S of crosscuts is considered satisfying: (i) to any crosscut $s \in S$ belongs one prime end of either bank; (ii) two distinct crosscuts have neither a point nor prime end belonging to them in common. Let $\zeta(z)$ map D conformally onto $|\Im\zeta| < \pi$ in such a manner that

$$\lim_{z\to p_1} \Re\zeta(z) = -\infty, \quad \lim_{z\to p_2} \Re\zeta(z) = +\infty.$$

Let $\xi_1(s) = \inf_{z \in s} \Re\zeta(z)$, $\xi_2(s) = \sup_{z \in s} \Re\zeta(z)$. Let $M(s_i, s_k)$ denote the modulus of the quadrilateral $Q(s_i, s_k)$ cut out of D by s_i, s_k where s_i separates s_k from p_1. The following theorem is established with the aid of methods due to Teichmüller [Deutsche Math. 3, 621–678 (1938)]: Let $p_1, s_i^*, s_i, s_k, s_k^*, p_2$ form a separating sequence. If

$$\min[M(s_i^*, s_i), M(s_k, s_k^*)] \geqq 2\pi,$$

then $\xi_2(s_k) - \xi_1(s_i) < M(s_i^*, s_k^*)$. *M. Heins.*

Beurling, Arne. An extension of the Riemann mapping theorem. Acta Math. 90, 117–130 (1953).

Let Φ denote a continuous positive bounded function defined in the finite plane. Let w_0 denote a given point of the finite plane. Functions f analytic in $|z| < 1$ satisfying the normalization: $f(0) = w_0$, $f'(0) > 0$ are considered. By A_Φ is meant the set of such f satisfying:

$$\limsup_{|z|\uparrow 1} \{|f'(z)| - \Phi(f(z))\} \leqq 0.$$

By B_Φ is meant the set of such f which are univalent and satisfy: $\lim\inf_{|z|\uparrow 1} \{|f'(z)| - \Phi(f(z))\} \geqq 0$. By C_Φ is meant the subset of B_Φ whose members f satisfy:

$$\lim_{|z|\uparrow 1} \{|f'(z)| - \Phi(f(z))\} = 0.$$

Let $A^* = \bigcup_{f \in A_\Phi} f[|z| < 1]$, $B^* = \bigcap_{f \in B_\Phi} f[|z| < 1]$. The following theorem is established. I. A^* is a simply-connected region and the normalized univalent conformal map f^* of $|z| < 1$ onto A^* is a member of C_Φ. II. B^* is a simply-connected region and the normalized univalent conformal map g^* of $|z| < 1$ onto B^* is a member of C_Φ. III. If $h \in C_\Phi$, then $B^* \subset h(|z| < 1) \subset A^*$. IV. If $\log \Phi^{-1}$ is subharmonic, $A^* = B^*$ and C_Φ has a unique member. The proofs involve the notion of a Schoenfliess type region, the concepts of extended union and reduced intersection of regions and compactness properties of the classes A_Φ, B_Φ, C_Φ.

M. Heins (Providence, R. I.).

Fourès, Léonce. Groupes fuchsiens et revêtements. Ann. Inst. Fourier Grenoble 4 (1952), 49–71 (1954).

Let G be a Fuchsian or Fuchsoid group of self-transformations of a disk C, not void of elliptic transformations. Consider a complex analytic mapping t of a Riemann surface T into another, S. Then (T, t) is called a regularly ramified covering of S at $M \in S$ if all components of $t^{-1}(M)$ are branch points of the same finite order [Fourès, Ann. Sci. Ecole Norm. Sup. (3) 69, 183–201 (1952); these Rev. 14, 550]. The author shows that there exists a Riemann surface R and a complex analytic mapping ψ of C onto R with the following properties: (1) ψ is invariant under the transformations in G; (2) (C, ψ) is a covering surface of R, regularly ramified at the images under ψ of the fixed points of the elliptic transformations in G; (3) the fundamental (Poincaré) group of R is isomorphic to the quotient group of G by the subgroup of the elliptic transformations in G.

The author proves, moreover, that these results are of topological nature. They remain valid if T, S, R are replaced by surfaces, t by a continuous mapping with isolated points $t^{-1}(M)$, and G by the "H-Fuchsoid" group of suitably restricted self-homeomorphisms of C. *L. Sario.*

Mori, Akira. An imbedding theorem on finite covering surfaces of the Riemann sphere. J. Math. Soc. Japan **5**, 263–268 (1953).

The following theorems are proved. Theorem 1. Let D be a subregion of a Riemann surface F such that the closure $\bar{D}$ of D is compact and that the boundary of D consists of a finite number of Jordan curves. Let $f(p)$ be a function defined and analytic on $\bar{D}$ (poles being admitted). Then, $\bar{D}$ can be imbedded in a closed Riemann surface D^* of the same genus as D in such a manner that $f(p)$ can be continued to a function defined and analytic on D^*. Theorem 3. Let F be an open Riemann surface, and G be a sub-region of F, whose complement $F-G$ is compact and is bounded by a finite number of Jordan curves. Let $f(p)$ be a given function analytic on the closure $\bar{G}$ of G. Suppose that F belongs to the class O_{HB}, and that $z=f(p)\neq\text{const.}$ is analytic on $\bar{G}$ and omits in G a set E of values z of positive logarithmic capacity. Then, $f(p)$ takes in G any value z at most a finite number of times N, and, along any continuous curve L: $p=p(t)$ $(0\leqq t<\infty)$ extending itself to the ideal boundary of F, $\lim f(p)$ exists. All these limiting values form a closed set of logarithmic capacity zero. Further, F belongs really to the class O_G. Theorem 1 is proved by combinatorial means and is in turn used to prove Theorem 3. *H. L. Royden* (Stanford, Calif.).

Andreian, Cabiria. Le théorème des disques pour les surfaces de Riemann normalement exhaustibles. Acad. Repub. Pop. Romåne. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 263–272 (1952). (Romanian. Russian and French summaries)

On étudie les surfaces de Riemann normalement exhaustibles, ayant un ordre de connexion arbitraire et l'on établit les résultats suivants, sur le nombre h des domaines simplement connexes et disjoints, qui ne sont couverts par aucun feuillet simple de la surface: A. Pour une surface de Riemann normalement exhaustible, dont le genre est zero, $h\leqq 3$. B. Si le genre g (resp. l'ordre de connexion c) et le nombre n des feuillets sont finis, $h\leqq 4+4(g-1)/n$ (resp. $h\leqq 2+2(c-2)/n$). C. Si le genre $g\geqq 1$ (resp. l'ordre de connexion c) est fini, mais la surface a une infinité de feuillets, $h\leqq 4$ (resp. $h\leqq 2$). D. Une condition nécessaire pour l'existence d'une infinité de domaines couverts par des feuillets multiples, est que le genre soit infini. Quelques exemples complètent l'exposé. (Author's summary.) *M. Heins* (Providence, R. I.).

Pfluger, Albert. Über das Typenproblem Riemann'scher Flachen. Comment. Math. Helv. **27** (1953), 346–356 (1954).

The paper is concerned with the class O_G of parabolic Riemann surfaces F, characterized by the nonexistence of Green's functions. For an exhaustion $\{F_n\}$ of F, the modulus μ_n of F_n-F_0 is defined by the conformal mapping of F_n-F_0 onto the annulus $1<|z|<\mu_n$ with suitably identified radial slits [Sario, C. R. Acad. Sci. Paris **230**, 269–271 (1950); these Rev. **11**, 342]. It is known that $F\in O_G$ if and only if $\lim\mu_n=\infty$. The author shows that the conformal moduli μ_n can be replaced by their algebraic counterparts, the modulus $\tilde{\mu}_n$ and the comodulus $\bar{\mu}_n$, with $\tilde{\mu}_n\to\infty$ being sufficient, $\bar{\mu}_n\to\infty$ necessary for $F\in O_G$. The reasoning is based on the use of weighted complexes on F [Royden, Trans. Amer. Math. Soc. **73**, 40–94 (1952); these Rev. **14**, 167].

To define $\tilde{\mu}_n$, $\bar{\mu}_n$, let K be a polyhedric representation of F with n-cells σ_k^n $(n=0, 1, 2)$, and let g_k be positive weights assigned to the edges σ_k^1. Given an exhaustion $\{k_n\}$ of K, let (k_0, k_n) be the "annulus" between the boundaries of k_0 and k_n. Consider the class $(X^1)_n$ of 1-forms $X^1=\sum x_k\sigma_k^1$ on K with the boundary $\partial X^1=1$ on $\sum\sigma_k^0\subset k_0$, and $\partial X^1=0$ at the vertices interior to (k_0, k_n). Then the modulus of (k_0, k_n) is defined as $\tilde{\mu}_n=\min\sum_1^\infty g_k^{-1}x_k^2$ in $(X^1)_n$. The comodulus $\bar{\mu}_n$ is obtained in an analogous manner, using cofunctions and coboundaries.

The author first shows that the boundedness of $\tilde{\mu}_n$ characterizes, in a specified sense, the type of the complex (K, g). The parabolic type of (K, g) implies, in turn, that $F\in O_G$, the hyperbolic cotype of (K, g) that F non-$\in O_G$. In special cases, the type and the cotype coincide. *L. Sario* (Cambridge, Mass.).

Radojčić, M. Sur le problème des types des surfaces de Riemann. Srpska Akad. Nauka. Zbornik Radova **35**. Mat. Inst. **3**, 15–28 (1953). (Serbo-Croatian. French summary)

"Aperçu élémentaire des critères concernant les types des surfaces de Riemann, sans prétendre à une liste complète".

Remmert, Reinhold, und Stein, Karl. Über die wesentlichen Singularitäten analytischer Mengen. Math. Ann. **126**, 263–306 (1953).

Soit M_D un ensemble analytique défini dans un domaine D de l'espace $C^n(z_1, \cdots, z_n)$. Un point frontière P de D est dit point ordinaire pour M_D s'il existe un voisinage $U(P)$ et un ensemble analytique M_1 défini dans $U(P)$ et tel que $M_D\cap U(P)\subset M_1\cap U(P)$; M_1 est le prolongement analytique de M_D dans $U(P)$. On notera que si $M_D\cap U(P)$ est vide, P est point ordinaire de M_D. Si le prolongement est impossible en P, P est dit point singulier essentiel pour M_D et appartient alors à l'adhérence $\bar{M}_D$. Soit F^k un ensemble analytique de dimension $k<n$ (dimension signifie dimension complexe), globalement irréductible dans D; soit M^k un ensemble analytique dans $D-F^k$ qui a en tous ses points la dimension k. Alors, ou bien M^k a tout point de F^k comme point singulier essentiel, auquel cas on a $F^k\subset\bar{M}^k$, ou bien M^k est prolongé par son adhérence $\bar{M}^k$ en un ensemble analytique dans D, de dimension k en tous ses points. Tel est l'énoncé principal du mémoire; sa démonstration utilise la même propriété établie par P. Thullen dans le cas particulier $k=n-1$ [Math. Ann. **111**, 137–157 (1935)]. Elle est précédée dans le mémoire d'une étude approfondie des ensembles analytiques et de leur répresentation locale obtenue en annulant des polynomes distingués, étude qui précise la notion de dimension complexe d'un ensemble analytique.

I) On définit la dimension de l'ensemble analytique M en $P\in M$ par $\dim_P M=n-k$, où k est la dimension complexe maxima des plans analytiques π^k par P tels que P soit point isolé de $\pi^k\cap M$. On généralise alors un théorème classique de Weierstrass: si $P_0(z_j^0, 1\leqq j\leqq n)$ appartient à M et est point isolé de $\pi^{n-k}\cap M$, où π^{n-k} est le plan $z_1=z_1^0, \cdots, z_k=z_k^0$, $1\leqq k<n$, on peut trouver un polycylindre $U(P_0)=E[|z_j-z_j^0|<\epsilon_j, 1\leqq j\leqq n]$ et un ensemble analytique M' dans $U(P_0)$, tel que $M\cap U(P_0)\subset M'$, M' étant défini en annulant $n-k$ polynomes distingués $\omega_{k+s}[z_{k+s}; z_1, \cdots, z_k]$, $1\leqq s\leqq n-k$, de centre projection ζ_0 de P_0 sur le sous espace $C^k(z_1, \cdots, z_k)$, auxquels s'ajoutent en

général des relations $\phi_i(z_1, \cdots, z_k)=0$ entre les variables non distinguées, les ϕ_i étant holomorphes pour $\zeta=(z_1, \cdots, z_k)$ dans $Z^k=E[z_p-z_p^0<\epsilon_p, 1\leq p\leq k]$. Les ω_{k+s} n'ont pas de facteur multiple en P_0 et leurs racines $z_{k+s}(z_1, \cdots, z_k)$ demeurent dans $|z_{k+s}-z_k^0|<\epsilon_{k+s}$ pour $\zeta \in Z^k$.

On améliore la conclusion si on apporte aux hypothèses les précisions suivantes. a) Si $\dim_{P_0} M=k$, les ϕ_i disparaissent; si $\zeta \in Z^k$ est pris hors des zéros W des discriminants des ω_{k+s}, il existe au moins un point $P \in M\cap U(P_0)$ se projetant en ζ et possédant un voisinage dans lequel M coïncide avec l'ensemble M' précédent. b) Si M est de dimension k en P_0 et aux points voisins de P_0, $\omega_{k+s}=0$ définit exactement la projection de $M\cap U(P_0)$ sur $C^{k+1}(z_{k+s}, z_1, \cdots, z_k)$ et tout point $P\subset M'\cap U(P_0)$ qui ne se projette pas sur W appartient à M.

Pour les ensembles M' définis par $\omega_{k+s}=0$, il existe des voisinages, polycylindriques $V(P_0)$ dans lesquels $M\cap V(P_0)$ se décompose en une somme d'ensembles M_λ' irréductibles dans $V(P_0)$, dont chacun est l'adhérence d'une composante connexe M_λ^* formée de points dont aucun ne se projette sur W (ce sont donc des points réguliers de M' au sens habituel); les M_λ^* sont sans point commun.

On est alors en mesure d'établir que la dimension $k(P)$ d'un ensemble analytique en P est invariante par les transformations analytiques complexes biunivoque et regulières au voisinage de P; $k(P)$ est une fonction semi-continue supérieurement de P et, plus précisément, on a
$$k(P_0)=\lim\sup k(P_n)$$
pour $P_n\to P_0$, $P_n\neq P_0$. Si M est irréductible en P_0 et si $\dim_{P_0} M=k$, M est de dimension k aux points de M dans un voisinage de P_0; il existe alors un voisinage $U(P_0)$ et une sous variété $N\subset U(P_0)\cap M$ définie dans $U(P_0)$ et de dimension $k-1$ au plus contenant tous les points non réguliers de M dans $U(P_0)$.

Au point de vue global un ensemble analytique M_D est décomposable en une somme de sous ensembles analytique irréductibles dans D; on montre que sur chacun d'eux la dimension est constante. La décomposition s'interprète simplement en considérant l'espace analytique $R(M_D)$ engendré par les germes irréductibles de M_D [cf. H. Cartan, Ann. Sci. Ecole Norm. Sup. (3) 61, 149–197 (1944); ces Rev. 7, 290]; $R(M_D)$ s'obtient en identifiant P, g_P et Q, g_Q' si $P=Q \in M_D$ et si les germes g_P, g_Q' représentent le même des germes irréductibles de M en P: les composantes irréductibles de M_D sont alors les projections des différentes composantes connexes de l'espace $R(M_D)$.

II) On en vient alors à la démonstration du théorème principal énoncé au début. Signalons qu'elle a été réexposée depuis par le second des auteurs [Séminaire de l'Ecole Norm. Sup., 1953–1954, exposés 13 et 14] sous une forme condensée et simplifiée grâce à un recours plus systematique à l'espace R des germes et à l'emploi de la proposition suivante, extension donnée par H. Cartan d'un énoncé de T. Radó. Soit Y un espace analytique et $f(y)$ une fonction à valeurs complexes, continue dans Y, holomorphe en tous les points de Y ou $f(y)\neq 0$. Alors f est holomorphe partout dans Y. Les points essentiels du mémoire sont: soit D un domaine borné, π^k un plan analytique à k dimensions $(0\leq k\leq n-1)$, M un ensemble analytique dans $D-\pi^k$, qui n'est pas de dimension zéro en tous ses points; alors M a des points dans tout voisinage de la frontière de D; si M est de dimension k en tous ses points et si π^k est défini par $z_i=0$ $(k+1\leq i\leq n)$, contenant l'origine, il existe un plan π^{n-k} par celle-ci qui coupe π^k et M en des points isolés. Si D est le polycylindre: $\zeta=(z_1, \cdots, z_k)$, $\eta=(z_{k+1}, \cdots, z_n)$, $|\zeta|=\sup|z_i|$, $1\leq i\leq k$, $|\zeta|<1$, $|\eta|<1$, L^k étant la partie du plan $\eta=0$ contenue dans D, alors si M^k est défini dans $D-L^k$, et de dimension k en tous ses points, l'ensemble des points de L^k ordinaires par rapport à M^k qui est relativement ouvert dans L^k, est aussi fermé dans L^k. Revenant au théorème principal, on voit que s'il existe sur F^k un point ordinaire pour M^k, tous les points réguliers de F^k le sont aussi; les autres forment un ensemble F' de dimension $k-1$ au plus dont on se débarrasse en démontrant par recurrence sur l le théorème suivant. Si F_D^l est un ensemble analytique dans D $(0\leq l<n-1)$, et si M^k est un ensemble dans $D-F_D^l$, qui a en tous ses points la dimension k $(1<k<n)$, alors l'adhérence $\bar{M}^k$ de M^k prolonge analytiquement M^k dans tout D et possède la dimension k en tous ses points.

Divers corollaires, notamment la démonstration d'une condition suffisante pour que $P \in \bar{M}^k$ soit point ordinaire de M^k, sont donnés à la suite du théorème principal, ainsi qu'une application aux ensembles analytiques de l'espace projectif P^n; ces ensembles s'identifient aux cônes analytiques (ensembles de droites complexes issus d'un sommet) de l'espace C^{n+1}. Un tel cône K est nécessairement défini en annulant des polynomes homogènes, d'où résulte très simplement le théorème de Chow qu'un ensemble analytique dans P^n est algébrique; s'il est de dimension $n-1$ en tous ses points, on l'obtient en annulant un seul polynome homogène. *P. Lelong* (Lille).

Rothstein, Wolfgang. Zur Theorie der Singularitäten analytischer Funktionen und Flächen. Math. Ann. **126**, 221–238 (1953).

Verf. versucht den in der Funktionentheorie einer komplexen Veränderlichen entwickelten Begriff der harmonischen Nullmenge auch in der Theorie analytischer Funktionen mehrerer komplexen Veränderlichen fruchtbar zu machen. Zunächst werden Mengen untersucht ("H-Mengen"), die in gewissem Sinne ohne Einfluss auf die Bildung von Regularitätshüllen sind, d.h. genauer niederdimensionale Mengen M des Raumes R^4 mit der Eigenschaft, dass bei beliebigem Gebiete G für die Regularitätshüllen die Beziehung $\mathfrak{H}(G-M)\supseteq\mathfrak{H}(G)-M$ gilt. Als wichtigstes Beispiel gibt Verf. das der "B-Mengen" an, wobei M eine B-Menge genannt wird, falls es zu jedem Punkt P aus M eine Kugel U und eine in $U-M$ reguläre biharmonische Funktion $b(w, z)$ gibt, sodass für $R \in M$, $Q_n\to R$, $Q_n \in U-M$ immer $\lim b(Q_n)=+\infty$. Eine besondere Klasse von B-Mengen sind die, für welche es zu jedem P aus M ein U und eine in U reguläre Funktion $\varphi(w, z)$ gibt, die auf M nur Werte aus einer harmonischen Nullmenge annimmt; M besteht dann aus analytischen Flächenstücken. (Für die Fragestellung als solche wäre es von Interesse, einerseits den Begriff der H-Mengen für 3-dimensionale Mengen—welche gnügend kleine Gebiete in getrennte Teile zerlegen—zu präzisieren und andererseits konkrete Beispiele von Mengen von niederer als dritter Dimension, die keine H-Mengen sind, aufzustellen.)

Die Benutzung harmonischer Nullmengen führt weiter zu Aussagen vom Typus des Hartogs'schen Hauptsatzes und zu einer Verschärfung sowohl eines Satzes von P. Thullen über die Singularitäten analytischer Flächen [Math. Ann. 111, 137–157 (1935)] wie auch einer in der oben referierten Arbeit bewiesenen Ausdehung dieses Satzes auf niederdimensionale analytische Flächen des R^{2n}. Schliesslich wird ein Analogon zum Hartogs'schen Hauptsatze für $(2n-2)$-dimensionale Flächen des R^{2n} $(n\geq 3)$ angegeben. *P. Thullen* (Genf).

Hua, Lo-Kèn. On the theory of functions of several complex variables. I. On a complete orthonormal system in the hyperbolic space of rectangular matrices. Doklady Akad. Nauk SSSR (N.S.) 93, 775–777 (1953). (Russian)

Let $Z=(x_{jk}+iy_{jk})$ be an $n\times n$ matrix with complex elements. Each Z has a representation point $(x_{jk};y_{jk})$ $(j,k=1,2,\cdots,n)$ in $2n^2$-dimensional Euclidean space E. Let R be the part of E corresponding to matrices Z for which $I-Z\bar{Z}'$ is positive definite. To every set (f) of integers $f_1\geq f_2\geq\cdots\geq f_n\geq 0$ there belongs an irreducible representation of the general linear group $\mathrm{GL}(n)$ mapping Z on $(a_{jk}^{(f)}(Z))$. The degree $N(f)$ of this representation can be calculated explicitly [see Weyl, Classical groups, Princeton, 1939; these Rev. 1, 42]. If

$$n^2(f)=\pi^2\prod_{j=1}^{n}(f_j+n-j)!/N(f)\prod_{j=1}^{n}(f_j+2n-j)!,$$

then the functions $\{a_{jk}^{(f)}(Z)/n(f)\}$ where j, k and (f) vary through all possible values form a complete orthonormal set in R. The Fourier kernel of this set is

$$K(Z,W)=\sum a_{jk}^{(f)}(Z)a_{jk}^{(f)}(\overline{W})/n^2(f)=C|I-Z\overline{W}'|^{-2n},$$

where

$$C=(2n-1)!(2n-2)!\cdots 2!1!/\pi^{n^2}[(n-1)!(n-2)!\cdots 2!1!]^2.$$

If L is the subspace of unitary matrices of R, then the $(a_{jk}^{(f)}(Z))$, after re-normalization, are a complete orthonormal set in L. Computing their Fourier kernel leads to the Cauchy-formula

$$f(Z)=k\int_L f(U)|I-Z\bar{U}'|^{-n}(dU),$$

where (dU) is the volume element of L. These results can be extended to the space of $m\times n$ matrices.

W. H. J. Fuchs (Ithaca, N. Y.).

Hua, Lo-Kèn. On the theory of functions of several complex variables. II. A complete orthonormal system in the hyperbolic space of hyperspheres. Doklady Akad. Nauk SSSR (N.S.) 93, 983–984 (1953). (Russian)

Let $X=(x_1,\cdots,x_n)$ be a vector of length 1 with real components. The orthogonal matrices operating on X induce a representation of the n-dimensional rotation group in the space of forms of degree m in the x_j. There are $[m/2]$ invariant subspaces Tl each one with a base of the form

$$(XX')^l\phi_{m-2l}^{(i)}(X)\quad(1\leq i\leq N_{m-2l})$$

where the ϕ's are polynomials such that

$$\int_S\phi_r^{(i)}(X)\phi_s^{(j)}(X)d\sigma=\delta_j^i\delta_s^r$$

(integration over the unit sphere S: $XX'=1$).

If $Z=(z_1,z_2,\cdots,z_n)$, $z_j=x_j+iy_j$, and if R is the domain $ZZ'<1$, $|ZZ'|^2+1<2Z\bar{Z}'$, then the functions

$$(ZZ')^l\phi_{m-2l}^{(i)}(Z)\quad(m=1,2,\cdots;0\leq l\leq m/2;1\leq i\leq N_{m-2l})$$

form a complete orthogonal set of functions in R, where the volume element is $\prod_{j=1}^n dx_jdy_j$. These functions are also orthogonal on the set L:

$$Z=W=e^{i\phi}X=(e^{i\phi}x_1,e^{i\phi}x_2,\cdots,e^{i\phi}x_n)\quad(0\leq\phi<2\pi;X\varepsilon S).$$

By computing their Fourier kernel it is found that for $f(Z)$ analytic in R

$$f(Z)=(2\pi)^{-1}\int_0^{2\pi}dy\int_S f(W)(1-2Z\overline{W}'+ZZ'\overline{WW'})^{-n/2}e^{i\phi}d\sigma.$$

It is pointed out that the analogue of the 3-circle theorem is valid for the sets $L(r)$: $Z=re^{i\phi}X$ (r real): If

$$F(r)=\left(\int_{L(r)}|f(Z)|^p(dZ)\right)^{1/p},$$

then $\log F$ is a convex function of $\log r$.

W. H. J. Fuchs (Ithaca, N. Y.).

Theory of Series

Knopp, Konrad. Folgenräume und Limitierungsverfahren. Ein Bericht über Tübinger Ergebnisse. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 11 (1952), 269–298 (1953).

Expository lecture given in March 1952 at a meeting held in conjunction with the first meeting of the International Mathematical Union.

Peyerimhoff, Alexander. Summierbarkeitsfaktoren für absolut Cesàro-summierbare Reihen. Math. Z. 59, 417–424 (1954).

The author gives a complete solution of a factor-sequence problem involving absolute Cesàro evaluability of series. Let α and β be nonnegative real numbers. The problem is to characterize those sequences ϵ_n for which $\sum\epsilon_na_n$ is evaluable $|C_\beta|$ whenever $\sum a_n$ is evaluable $|C_\alpha|$. The conditions are (i) $\epsilon_n=O(1)$ in case $\beta>\alpha$ and $\epsilon_n=O(n^{\beta-\alpha})$ in case $\beta\leq\alpha$ and (ii) $\Delta^\alpha(\epsilon_n)=O(n^{-\alpha})$.

R. P. Agnew.

Jurkat, W., und Peyerimhoff, A. Der Satz von Fatou-Riesz und der Riemannsche Lokalisationssatz bei absoluter Konvergenz. Arch. Math. 4, 285–297 (1953).

The authors give seven theorems on absolute convergence of power series and trigonometric series. These arise as a result of seeking theorems on absolute convergence analogous to theorems of M. Riesz (if $\sum a_nz^n$ is regular at $z=1$ and $a_n\to 0$, then $\sum a_n$ converges) and Riemann (the localization principle) on ordinary convergence. The theorems involve differences Δ^pa_n of sequences of coefficients, and most of them have statements too long for presentation here. The first says that if $\sum a_nz^n$ is regular at $z=1$ and if, for some nonnegative integer p, $\sum_{n=0}^\infty|\Delta^pa_n-\Delta^pa_{n+1}|<\infty$, then $\sum|a_n|<\infty$.

R. P. Agnew (Ithaca, N. Y.).

Harington, C. F., and Hyslop, J. M. An analogue for strong summability of Abel's summability method. Proc. Edinburgh Math. Soc. (2) 9, 28–34 (1953).

Let $k>-1$ and let $c_0^{(k)},c_1^{(k)},\cdots$ denote the Cesàro transform of order k of $\sum a_n$. The series is evaluable $[C,k,p]$, or strongly evaluable (C,k) with exponent p, to s if

$$\sum_{j=0}^{n}|c_j^{(k-1)}-s|^p=o(n)$$

as $n\to\infty$. The series is evaluable $[A,p]$ to s if the series in $f(u)=\sum a_ne^{-n/u}$ converges for $u>0$, if $f(u)\to s$ as $u\to\infty$, and if

$$\int_0^t|uf'(u)|^pdu=o(t)$$

as $t\to\infty$. It is shown that if $k\geq 0$ and $p\geq 1$, then evaluability $[C,k,p]$ implies evaluability $[A,p]$, and that if $p>q\geq 1$, then evaluability $[A,p]$ implies evaluability $[A,q]$.

R. P. Agnew (Ithaca, N. Y.).

Lorentz, G. G., and Robinson, A. Core-consistency and total inclusion for methods of summability. Canadian J. Math. 6, 27–34 (1954).

Let A and B be regular matrices with elements a_{nk} and b_{nk}. Suppose that, A is "stronger" than B in the sense that each complex sequence has the core of its A-transform contained in the core of its B-transform. It is of course too much to expect that these hypotheses imply existence of a regular matrix C such that $A=CB$, that is, $CB-A=0$. In case $b_{nk}\geqq 0$, the authors succeed in proving what seems to be the most that could be expected, namely, that there is a regular matrix C with $c_{nk}\geqq 0$ such that $CB-A$ is a matrix D for which

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}|d_{nk}|=0.$$

The proof involves theorems about convex sets in Banach spaces. Related results and applications are given.

R. P. Agnew (Ithaca, N. Y.).

Petersen, G. M. Methods of summation. Pacific J. Math. 4, 73–77 (1954).

The Rogosinski-Bernstein method (B^h) has been treated by Karamata [Mat. Sbornik N.S. **21**(63), 13–24 (1947); Math. Z. **52**, 305–306 (1949); these Rev. **9**, 140; **11**, 347], Agnew [Ann. of Math. (2) **56**, 537–559 (1952); these Rev. **14**, 368] and the author [Canadian J. Math. **4**, 445–454 (1952); these Rev. **14**, 368] for $h>0$. Now the author considers the case $h<0$ and constructs examples after a method of W. A. Hurwitz [Proc. London Math. Soc. (2) **26**, 231–248 (1927)] to show that for $h<0$ the method (B^h) sums a series not summable (C). In the second part of the paper the author proves: If $[1-(n+3)^{-1}]S_n+(n+3)^{-1}S_n\to\sigma$, then $S_n=C(-1)^{n-1}(n+1)!+$a convergent sequence.

K. Zeller (Philadelphia, Pa.).

Zeller, K. Approximation in Wirkfeldern von Summierungsverfahren. Arch. Math. 4, 425–431 (1953).

Let C be the set of all series $\bar{a}$ with terms a_k, $k=0, 1, \cdots$, summable by a matrix A, $\bar{e}_k$ series with terms $0, 0, \cdots, 0, 1, 0, \cdots$ (1 in kth position). In a natural topology, C is a Banach space or space of type (F). Mazur [Studia Math. **2**, 40–50 (1930)] proved that $\bar{a}$ belongs to the linear closure of the $\bar{e}_k$ if A is the Cesàro or the Euler-Knopp method. This is established here in a direct way for the general Euler-Knopp and the Borel methods. For the Cesàro method C_p, $p>0$, $\bar{a}$ is the limit in norm of the C_p-transform of the series $\sum a_k\bar{e}_k$.

G. G. Lorentz (Detroit, Mich.).

Zeller, Karl. Matrixtransformationen von Folgenräumen. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 12 (1953), 340–346 (1954).

Matrix transformations of one sequence space $\mathfrak{E}$ into another, $\mathfrak{F}$, are considered, where $\mathfrak{E}$ and $\mathfrak{F}$ are BK-spaces. These are Banach spaces of sequences $s=\{s_k\}$ for which the mapping $s\to s_k$ is linear and continuous. Examples are: (i) S_N, the set of null sequences $\{s_k\}$ with $\lim s_k=0$ and norm $\sup|s_k|$; (ii) T_A, the set of $\{s_k\}$ with $\sum|s_k|<\infty$ and norm $\sum|s_k|$. If all finite sequences belong to $\mathfrak{E}$, and if all elements in $\mathfrak{E}$ are such that $s^{(p)}=\{s_0, s_1, \cdots, s_p, 0, 0, 0, \cdots\}\to s$ as $p\to\infty$, then $\mathfrak{E}$ is said to be sectionally-convergent.

The following results are established. (1) If $A=(a_{n,k})$ transforms the BK-space $\mathfrak{E}$ into the BK-space $\mathfrak{F}$, then this transformation is continuous, i.e., $\|As\|_{\mathfrak{F}}\leqq p\|s\|_{\mathfrak{E}}$ for every $s\in\mathfrak{E}$. (2) Let $\mathfrak{E}$ and $\mathfrak{F}$ be BK-spaces, and let $\mathfrak{E}$ be sectionally-convergent. Also, for every finite sequence s, let $As\in\mathfrak{F}$, and (A) $\|As\|_{\mathfrak{F}}\leqq p\|s\|_{\mathfrak{E}}$ (s finite). Then $\mathfrak{E}$ is transformed by A into $\mathfrak{F}$. (A) can be simplified in special cases. Thus, when $\mathfrak{E}=T_A$ (defined above), (A) is equivalent to $\|a^{(k)}\|_{\mathfrak{F}}\leqq\sigma$ ($k=0, 1, 2, \cdots$), $a^{(k)}=\{a_{n,k}\}=Ae^{(k)}$, $e^{(k)}$ being the fundamental unit sequence with 1 for its kth element and all other elements 0. Again, when $\mathfrak{E}=S_N$ (defined above), (A) is equivalent to $\|\sum_{k\in K}a^{(k)}\|_{\mathfrak{F}}\leqq\sigma$, where K is an arbitrary finite subset of the positive integers. Various conditions on $A=(a_{n,k})$ are then given which are equivalent to the property "A maps $\mathfrak{E}$ into $\mathfrak{F}$" in the cases where $\mathfrak{E}=\mathfrak{F}=T_A$; $\mathfrak{E}=T_A$, $\mathfrak{F}=S_N$; $\mathfrak{E}=\mathfrak{F}=S_N$; $\mathfrak{E}=S_N$, $\mathfrak{F}=T_A$. Finally, the inequalities in (1) and (2) above can be restated as $|f(As)|\leqq\sigma\|f\|\,\|s\|$ ($f\in\mathfrak{F}_1^*$), where $\mathfrak{F}_1^*$ is a suitable subset of the dual (conjugate) space of $\mathfrak{F}$.

R. G. Cooke.

Zeller, K. Über die Darstellbarkeit von Limitierungsverfahren mittels Matrixtransformationen. Math. Z. 59, 271–277 (1953).

The author deals with the question whether certain methods of assigning generalized limits to sequences are equivalent with matrix methods. In particular, a sequence $\{x_n\}$ is $\bar{C}_1^q$-limitable to x provided $\sum_{k=0}^{n}|x_k-x|^q=o(n)$. Kuttner [J. London Math. Soc. **21**, 118–122 (1946); these Rev. **8**, 375] showed that, for $0<q<1$, the method $\bar{C}_1^q$ is not equivalent to any matrix method; the author shortens Kuttner's proof, and shows that for each $q\geqq 1$ the method $\bar{C}_1^q$ is equivalent to some row-finite matrix method. With each matrix A the author associates the set $\mathfrak{S}_NA$ of sequences whose A-transforms are null-sequences. Applying the methods of his earlier work on FK-spaces [Math. Z. **55**, 55–70 (1951); these Rev. **13**, 934], he shows that if (1) the matrix methods A^i ($i=0, 1, \cdots$) are regular and perfect; (2) $\mathfrak{S}_NA^{(i)}\supset\mathfrak{S}_NA^{(i+1)}$ for each i; (3) for every unbounded increasing sequence $\{\delta_k\}$ and every j there exists a sequence $\{x_k\}$, contained in $\mathfrak{S}_NA^{(j)}$ but not in $\bigcap\mathfrak{S}_NA^{(i)}$, and with $|x_k|<\delta_k$; then there exists no matrix A with the property that $\mathfrak{S}_NA=\bigcap\mathfrak{S}_NA^{(i)}$.

G. Piranian (Ann Arbor, Mich.).

Jakimovski, Amnon. On a Tauberian theorem by O. Szász. Proc. Amer. Math. Soc. 5, 67–70 (1954).

Généralisation d'un résultat obtenu par Rényi [Acta Univ. Szeged. Sect. Sci. Math. **11**, 119–123 (1946); ces Rev. **8**, 147], repris par O. Szász [Pacific J. Math. **1**, 117–125 (1951); ces Rev. **13**, 227], une condition nécessaire et suffisante de convergence de la série réelle $\sum_0^\infty a_k$ est qu'elle soit sommable $(A, 1)$ et que

$$\liminf_{m\to\infty}\frac{1}{m+1}\left(\sum_{m+1}^{n}ka_k\right)\geqq 0,\quad \frac{n}{m}\to 1,\quad n>m$$

ou que

$$\liminf_{m\to\infty}\left\{\frac{1}{n}\sum_{0}^{n}ka_k-\frac{1}{m}\sum_{0}^{m}ka_k\right\}\geqq 0,\quad \frac{n}{m}\to 1,\quad n>m.$$

Des conditions de convergence sont indiquées lorsqu'on remplace le procédé $(A, 1)$ par le procédé (B).

M. Zamansky (Paris).

Ščeglov, M. P. On bounded sequences. Doklady Akad. Nauk SSSR (N.S.) 90, 145–147 (1953). (Russian)

Let $s_0, s_1, \cdots$ be a real bounded sequence. Let $\underline{D}_0=\liminf s_n$ and $\bar{D}_0=\limsup s_n$. Let $\underline{D}_1=\liminf\sigma_n$ and $\bar{D}_1=\limsup\sigma_n$ where $\sigma_n=(s_0+s_1+\cdots+s_n)/(n+1)$. Let $\underline{D}_2=\liminf\phi(u)$ and $\bar{D}_2=\limsup\phi(u)$ where $u\to\infty$ and $\phi(u)=u^{-1}\sum_{k=0}^{\infty}s_ke^{-k/u}$. It is well known that $\underline{D}_0\leqq\underline{D}_1\leqq\underline{D}_2\leqq\bar{D}_2\leqq\bar{D}_1\leqq\bar{D}_0$. Some additional results, including the following, are given. If $\bar{D}_1=\bar{D}_0$, then $\bar{D}_2=\bar{D}_0$. The number e is the least constant

such that

$$(\bar{D}_1-\underline{D}_1)\leqq\varepsilon(\bar{D}_2-\underline{D}_2)$$

whenever s_n is bounded. *R. P. Agnew* (Ithaca, N. Y.).

Stuloff, N. Ein Beitrag zur Theorie spezieller Dirichletscher Reihen. Math. Z. 59, 339–355 (1954).

The author discusses the representation of a function by a Dirichlet series $\sum a_n e^{-\lambda_n s}$ with nonnegative coefficients a_n and exponents λ_n. He was apparently unaware that the problem had been discussed before by Kaluza [same Z. 28, 203–215 (1928)] and by Widder [Trans. Amer. Math. Soc. 39, 244–298 (1936)]; his results all seem to be consequences of Widder's. *R. P. Boas, Jr.* (Evanston, Ill.).

Fourier Series and Generalizations, Integral Transforms

***Bray, Hubert E. Convergence of Fourier series.** Rice Inst. Pamphlet. Special Issue, Nov. 1953, pp. 1–30. The Rice Institute, Houston, Texas, 1953.

Let $\Phi(t)$ be L-integrable over $\langle 0, \pi\rangle$ and let there

$$Q(t)=t^{-1}\int_0^t\Phi(\tau)d\tau \quad\text{and}\quad Q_1(t)=\int_0^t Q(\tau)d\tau.$$

Let also $I_\nu(\lambda, \mu)=\int_\lambda^\mu\Phi(t)t^{-1}e^{i\nu t}dt$ where $0<\pi/\nu\leqq\lambda<\mu\leqq\pi$. The main result of the paper is as follows: If (*) $Q(t)=o(1)$ as $t\rightarrow+0$, then the following two conditions imply each other: (A) $I_\nu(\lambda, \mu)\rightarrow 0$ as $\nu\rightarrow\infty$ (uniformly for $\pi/\nu\leqq\lambda<\mu\leqq\pi$), and (B) $S(Q_1, \lambda, h, 2k)-\int_0^{2kh}Q_1(\lambda+t)dt=o(h^2)$ as $h\rightarrow+0$ (uniformly for all intervals $\langle\lambda, \lambda+2kh\rangle$ in $\langle 0, \pi\rangle$). Here S in (B) is the Simpson approximation sum (distance h) for the integral $\int_0^{2kh}$. In the theory of Fourier series, where $\Phi(t)=f(x_0+t)+f(x_0-t)-2f(x_0)$, the condition (*) is satisfied p.p., and it is familiar that (*) together with (A) certainly implies convergence of the Fourier series at x_0 to the value $f(x_0)$. In fact, (A) is of similar relevance for the convergence of the conjugate series and for uniform convergence of both series. (*) and (B) together can be considered as a convergence test in the theory of Fourier series. It is shown that this test is not included in the familiar Lebesgue test, and the author lays much stress on this fact. However, since (B) has been proved equivalent to (A), this seems to the reviewer not very surprising.

W. W. Rogosinski (Newcastle-upon-Tyne).

***Nash, John P. Uniform convergence of Fourier series.** Rice Inst. Pamphlet. Special Issue, Nov. 1953, pp. 31–57. The Rice Institute, Houston, Texas, 1953.

Let $0<\Phi(t)\rightarrow a$ as $t(>0)\rightarrow\infty$. An L-integrable function $f(x)$ of period 2π is said to be of class Φ if

$$\int_a^b f(x+t)\cos ntdt=O(\Phi^{-1}(n))$$

uniformly for all x and all $a<b$ such that $b-a\leqq 2\pi$. The case $\Phi(t)=t$ [functions of écart fini] was first discussed by Hadamard. It is easy to see that one must have $\Phi(u)=O(u)$ in any case. The main result is as follows: Let $t\leqq\psi(t)\leqq\log t$ and $\psi'(t)>k\psi(t)/t$ where $k>0$. If $f(x)$ is bounded and has Fourier coefficients $O(\psi^{-1}(n))$, then $f(x)$ is of class $\psi(t)/\log t$. Again, if $s_n(x)$ is the partial sum of the Fourier series of a continuous function $f(x)$ of class Φ where $\Phi'(t)=O(1)$, uniform estimates for $|f(x)-s_n(x)|$ are given involving the modulus of continuity of f. Finally, an example of a continuous function of class $t/(\log t)^{1/2}$ is given whose Fourier series converges uniformly. [If $\Phi(t)=t$ and f is continuous, the uniform convergence follows, of course, from Fejér's theorem. In fact, it is required only that the Fourier coefficients be $O(1/n)$; such a function would be, by the above result, of class $t/\log t$.] *W. W. Rogosinski.*

Korenblyum, B. I. On the convergence theory of Fourier series. Dopovidi Akad. Ukrain. RSR 1951, 320–323 (1951). (Ukrainian. Russian summary)

The author gives the following generalization of the Lebesgue-Gergen criterion for the convergence of Fourier series [Gergen, Quart. J. Math., Oxford Ser. 1, 252–275 (1930)]. Let $f(x)$ be periodic, with period 2π, even and integrable. The Fourier series of f converges to 0 at the point 0 provided

$$\int_0^\tau f(t)dt=o(\tau),$$

$$\lim_{k\to\infty}\limsup_{\tau\to 0}\int_{k\tau}^a t^{-1}\left|\sum_{n=0}^N(-1)^n\alpha_n f(t+n\tau)\right|dt,$$

where α_n $(n=0, 1, \cdots, N)$ are non-negative numbers not all equal to zero and satisfying the condition

$$\alpha_0+\alpha_2+\alpha_4+\cdots=\alpha_1+\alpha_3+\cdots$$

(a is any positive number). In Gergen's case $\alpha_n=\binom{N}{n}$.

A. Zygmund (Cambridge, England).

Alexits, G. Über den Einfluss der Struktur einer Funktion auf die Konvergenz fast überall ihrer Fourierreihe. Acta Math. Acad. Sci. Hungar. 4, 95–101 (1953). (Russian summary)

It is well known that the convergence of $\sum_1^\infty(a_n^2+b_n^2)\log n$ implies the convergence p.p. of the Fourier series of an $f(x)\in L^2$ with Fourier coefficients a_k, b_k. Let

$$\omega_2(\delta, f)=\sup_{|h|\leqq\delta}\left\{\int_0^{2\pi}[f(x+h)-f(x)]^2dx\right\}^{1/2}.$$

It is shown, by the argument usual in this problem, that the above condition is equivalent to $\omega_2(\delta, f)=O(1/(\lambda(1/\delta))^{1/2})$, where $\lambda(x)$ is a positive increasing function such that $\int_1^\infty(x\lambda(x))^{-1}dx<\infty$. *W. W. Rogosinski.*

***Vogel, Théodore. Les fonctions orthogonales dans les problèmes aux limites de la physique mathématique.** Centre d'Etudes Mathématiques en Vue des Applications. C. Physique mathématique, vol. II. Centre National de la Recherche Scientifique, Paris, 1953. 191 pp. (1 plate). 1200 frs.

The following survey of the content conveys an idea of the general direction of the book. In chapter 1 orthogonal functions and systems of differential equations are treated (function space, orthogonality, mean convergence, closure, perturbation problems, variational principles). The subject of chapter 2 is the study of the closure of some special sequences occurring in mathematical physics. Finally, in chapter 3 various applications are dealt with, like the Hamilton-Jacobi equation, Schrödinger's equation, wave equation, heat equation, and problems of forced vibrations.

G. Szegö (Stanford, Calif.).

Campbell, Robert. Sur les séries de Neumann de variables réelles. C. R. Acad. Sci. Paris 238, 983–984 (1954).

If $S_r(x)$ is the rth partial sum of the Neumann expansion of $f(x)$ into a series of Bessel functions, put

$$\tau_n(x)=\sum_1^n\alpha_r S_r(x)/\sum_1^n\alpha_r,$$

where $\alpha_r = \frac{1}{2}(r-1)$. An explicit formula for the error $r_n(x)-f(x)$ is obtained. *W. Rudin* (Rochester, N. Y.).

Doss, Raouf. Sur une nouvelle classe de fonctions presque-périodiques. C. R. Acad. Sci. Paris **238**, 317–318 (1954).

A real function $f(x)$ defined in $(-\infty, \infty)$ is called R-almost periodic if one can find two numbers ξ and M with the following property: For every $\epsilon>0$ there exist an integer n, a number $\delta>0$, and positive numbers $\pi_1, \cdots, \pi_m$, such that $|n^{-1}\sum_0^{n-1} f(x_l+l\xi)-M|<\epsilon$ whenever $|x_i-x_j|\leq\delta$ (mod π_k) ($i, j=0, 1, \cdots, n-1$; $k=1, 2, \cdots, m$). This definition generalizes a certain characterization of the R-integrable functions $f(x)$ with a given period. The following approximation theorem is stated without proof: A function $f(x)$ is R-almost periodic if and only if for every $\epsilon>0$ there exist trigonometric polynomials $p(x)$ and $q(x)$ such that $p(x)\leq f(x)\leq q(x)$ and $\mathfrak{M}\{q(x)-p(x)\}\leq\epsilon$. *B. Jessen.*

Horváth, Jean. Sur l'itération de la transformée de Hilbert d'une distribution complexe. C. R. Acad. Sci. Paris **237**, 1480–1482 (1953).

Let $z=x+iy$, $\zeta=\xi+i\eta$ denote points of the complex plane R^2 and let $K_n(z)=z^n/|z|^{n+2}$. For any $f(z)$ we set

$$T_n f = f * K_n = \iint_{R^2} f(\zeta)K_n(z-\zeta)d\xi d\eta.$$

Mihlin [Uspehi Matem. Nauk (N.S.) **3**, no. 3(25), 29–112 (1948); these Rev. **10**, 305] proved that if $f\in L^2$ the integral on the right exists in the l.i.m. sense, $T_n f\in L^2$ and the operations T_n, suitably normalized, form a group. The author proves the latter result for T_n in the sense of L. Schwartz's distributions. Both proofs use the method of Fourier transforms though in Mihlin's proof this is disguised by a different terminology. *A. Zygmund* (Cambridge, England).

Obreškov, N. Concerning solutions of some singular integral equations. Doklady Akad. Nauk SSSR (N.S.) **92**, 1117–1120 (1953). (Russian)

The author gives an inversion and a representation theorem for a generalized Laplace transform he has considered elsewhere [Bŭlgar. Akad. Nauk. Izvestiya Mat. Inst. **1**, 83–110 (1953); cf. these Rev. **15**, 119, for its connection with the work of other authors]. *R. P. Boas, Jr.*

Mayer-Kalkschmidt, Jörg. Singularitäten von Laplace-Integralen an der Summierbarkeitsabszisse. Arch. Math. **4**, 441–445 (1953).

Let $f(s)=\int_0^\infty e^{-st}a(t)dt$. If for some positive integer k the above integral has (C, k) summability axis b, if the right hand derivative of $f(s)$ at $s=b$ is real and non-negative, and if $t^{-k}\int_0^t e^{-bu}(t-u)^k a(u)du$ increases monotonely for large t, then $s=b$ is a singular point of $f(s)$.

I. I. Hirschman, Jr. (St. Louis, Mo.).

Rooney, P. G. An inversion and representation theory for the Laplace integral of abstractly-valued functions. Canadian J. Math. **6**, 190–209 (1954).

Laplace transforms of functions on $B_p([0, \infty], X)$, where X is a Banach space, and the notation is that of Hille, "Functional analysis and semi-groups" [Amer. Math. Soc. Colloq. Publ., v. 31, New York, 1948; these Rev. **9**, 594], are studied by means of the inversion operator

$$L_{k,\tau}f(\lambda)=(ke^{2k}/\pi\tau)\int_0^\infty \eta^{-\frac{1}{2}}\cos(2k\eta^{\frac{1}{2}})f(k(\eta+1)/\tau)d\eta.$$

If $f(\lambda)$ is the Laplace transform of a function $x(\tau)$ on $(0, \infty)$ to X, conditions are given in order that

$$\lim_{k\to\infty} L_{k,\tau}f(\lambda)=x(\tau),$$

and further conditions are given in order that $f(\lambda)$ should be such a transform with $x(\tau)$ in $B_p([0, \infty], X)$, $1\leq p$.

J. L. B. Cooper (Cardiff).

Rooney, P. G. Some remarks on Laplace's method. Trans. Roy. Soc. Canada. Sect. III. **47**, 29–34 (1953).

The author generalizes some formulae of Widder on the asymptotic behaviour (as $k\to\infty$) of integrals of the form: $\int_a^b \exp(kh(x))\phi(x)dx$. *S. Agmon* (Jerusalem).

Delange, Hubert. Sur certaines intégrales de Laplace. Bull. Sci. Math. (2) **77**, 141–168 (1953).

Detailed proofs are given to some general results on singularities of Laplace transforms announced previously [C. R. Acad. Sci. Paris **233**, 1413–1414 (1951); these Rev. **13**, 551]. *S. Agmon* (Jerusalem).

Polynomials, Polynomial Approximations

Freud, G. Über einen Satz von P. Erdös und P. Turán. Acta Math. Acad. Sci. Hungar. **4**, 255–266 (1953). (Russian summary)

Erdös and Turán have discussed [Ann. of Math. (2) **41**, 510–553 (1940); these Rev. **1**, 333] the difference of two consecutive zeros of the orthogonal polynomials $p_n(x)$ associated with a given weight function $w(x)$. The author extends these results to the zeros $x_\nu = x_\nu(n)$ of polynomials of the form $p_n(x)+Ap_{n-1}(x)+Bp_{n-2}(x)$, $B\leq 0$, assuming that these zeros are all simple and real. Let $0<m\leq w(x)\leq M$ in a sub-interval $[a, b]$ of $[-1, +1]$. We have then

$$c_1/n < x_{k+1,n}-x_{k,n} < c_2/n$$

provided $a+\epsilon\leq x_{k,n}$, $x_{k+1,n}\leq b-\epsilon$, $\epsilon>0$. Assuming that $(1-x^2)^{1/2}w(x)$ is bounded away from zero and infinity in $[-1, +1]$, similar inequalities can be obtained for all $\theta_{k,n}$ where $x_{k,n}=\cos\theta_{k,n}$. The Cotes numbers of the associated mechanical quadrature enter also in this discussion.

G. Szegö (Stanford, Calif.).

Freud, Géza. Über die absolute Konvergenz von orthogonalen Polynomreihen. Acta Math. Acad. Sci. Hungar. **4**, 127–135 (1953). (Russian summary)

Let $\{\Phi_n(x)\}$ be a complete orthonormal system over $\langle -1, 1\rangle$ with respect to the weight function $w(x)\geq 0$, and let $f(x)\sim\sum_1^\infty c_k\Phi_k(x)$ where $w^{1/2}f\in L^2$. A very simple argument proves that the two conditions (i) $\sum_1^n|\Phi_k^2(x)|=O(n)$ and (ii) $\sum_{n=1}^\infty\{n^{-2}\sum_{n+1}^\infty|c_k|^2\}^{1/2}$ together imply the absolute convergence of $\sum_1^\infty c_k\Phi_k(x)$ at x [for (ii) cf. S. B. Stečkin, Mat. Sbornik N.S. **29**(71), 225–232 (1951); these Rev. **13**, 229]. In the case where the $\Phi_n(x)$ are polynomials and $f(x)$ is continuous, (ii) can be replaced by $\sum_1^\infty n^{-1/2}\omega^*(n^{-1}, f)<\infty$ where $\omega^*(\delta, f)=\max_{|\theta_1-\theta_2|\leq\delta}|f(\cos\theta_1)-f(\cos\theta_2)|$. If $f(x)$ is of bounded variation and $w(x)\leq W(1-x^2)^{-1/2}$, then (ii) can be replaced by $\sum_1^\infty\{n^{-1}\omega^*(n^{-1}, f)\}^{1/2}<\infty$. [For condition (i) cf. G. Freud, Acta Math. Acad. Sci. Hungar. **3**, 83–88 (1952); these Rev. **14**, 467.] *W. W. Rogosinski.*

Freud, Géza. Über die Lebesgueschen Funktionen der Lagrangeschen Interpolation. Acta Math. Acad. Sci. Hungar. 4, 137–142 (1953). (Russian summary)

For every $n \geq 1$, the numbers $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ are given, and the $l_{k,n}(x)$ denote the associated basic polynomials in the Lagrange interpolation formula. The following theorem is proved. The integrable function $w(x)$ is non-negative in $\langle a, b\rangle$, and the $P_n(x)$ are the normalized orthogonal polynomials over $\langle a, b\rangle$ associated with the weight $w(x)$. If the $x_{k,n}$ are the roots of $P_n(x)$, and if $|P_n(x)| \leq K$ and $0 < m \leq w(x) \leq M$ holds in a subinterval $\langle c, d\rangle$, then $\sum_1^n |l_{k,n}(x)| = O(\log n)$ uniformly in every closed subinterval of $\langle c, d\rangle$. This result covers the known cases of the zeros of Tchebycheff and Jacobi polynomials [cf. G. Szegö, Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., v. 23, New York, 1939; these Rev. 1, 14]. As an application the uniform convergence of the Lagrange interpolation to a continuous function $f(x)$, in every closed subinterval of $\langle c, d\rangle$, is proved, provided that $f(x)$ satisfies the Lipschitz condition $|f(x_2) - f(x_1)| = o(|\log |x_2 - x_1||^{-1})$ uniformly in $\langle c, d\rangle$. *W. W. Rogosinski.*

Freud, Géza. On the Lebesgue functions of Lagrange interpolation. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 3, 563–568 (1953). (Hungarian)

Let $p(x) \geq 0$ be a weight function of the class L in the finite interval $[a, b]$, $\{p_n(x)\}$ the associated orthonormal polynomials. We consider the Lagrange interpolation on the zeros of $p_n(x)$ and the corresponding Lebesgue functions $\Lambda_n(x)$. The author proves the following theorems. I. Let $[\alpha, \beta]$ be in the interior of $[a, b]$ such that $p(x)$ is bounded away from zero and infinity on $[\alpha, \beta]$ and $|p_n(x)| \leq K$ in $[\alpha, \beta]$; then $\Lambda_n(x) = O(\log n)$ uniformly in every interval in the interior of $[\alpha, \beta]$. II. Let $f(x)$ be defined in $[a, b]$ and satisfy the condition $\omega(\delta) < o\{|\log \delta|^{-1}\}$. Let $p(x)$ satisfy the same conditions as in I. Then the interpolation polynomials defined above converge uniformly to $f(x)$ in every interval interior to $[\alpha, \beta]$.

Finally, an estimate of the error of the interpolation is obtained provided $f(x)$ has a derivative of given order in $[a, b]$ satisfying a Lipschitz condition. *G. Szegö.*

Alexits, Georg. Eine Bemerkung zur Konvergenzfrage des Lagrangeschen Interpolationsverfahrens. Acta Math. Acad. Sci. Hungar. 4, 233–236 (1953). (Russian summary)

Let $p(x) > 0$ be a weight function in the finite interval $[a, b]$, $\{p_n(x)\}$ the associated orthonormal polynomials. For the Lebesgue function $\Lambda_n(x)$ of the corresponding Lagrange interpolation we have the following useful inequality:

$$[\Lambda_n(x)]^2 \leq \int_a^b p(t)dt \sum_{k=0}^{n-1} [p_k(x)]^2.$$

As a consequence, the following results are obtained. (1) Let $p(x)$ vanish only on a zero set; the Lagrange polynomials of a given function $f(x)$ converge almost everywhere in $[a, b]$ provided $f(x)$ satisfies a Lipschitz condition of order α, $\alpha > \frac{1}{2}$. (2) Let $p(x) \geq m > 0$ in a sub-interval $[\alpha, \beta]$ of $[a, b]$, and let $f(x)$ satisfy the same condition as in (1). Then the Lagrange polynomials converge uniformly in $[\alpha, \beta]$. *G. Szegö* (Stanford, Calif.).

Aczél, J. Eine Bemerkung über die Charakterisierung der "Klassischen" orthogonalen Polynome. Acta Math. Acad. Sci. Hungar. 4, 315–321 (1953). (Russian summary)

The author deals with the following characterization of the system of the Jacobi, Laguerre, and Hermite polynomials, respectively. Let $\{u_n(x)\}$ be a sequence of functions satisfying the following conditions:

(a) $$Q_n(x)u_n'(x) = L_n(x)u_n(x)$$

where Q_n and L_n are polynomials of degree 2 and 1, respectively; (b) all $u_n(x)$ have at least two distinct (finite or infinite) zeros a, b in common; (c) there is a function $p(x)$ so that $u_n^{(n)}(x)/p(x) = R_n(x)$ is a polynomial of the precise degree n. Then $\{R_n\}$ coincides with one of the systems mentioned above according as $Q_n(x)$ has the degree 2, 1, 0. The weight function will be $p(x)$ and the interval of orthogonality coincides with $[a, b]$. *G. Szegö.*

Palamà, Giuseppe. Sulla derivata erresima di classici polinomi rispetto ai parametri. Boll. Un. Mat. Ital. (3) 8, 401–409 (1953).

The author studies the higher derivatives of the Laguerre polynomial $L_n(\alpha; x)$ with respect to α and represents them in terms of $L_\nu(\alpha; x)$, $0 \leq \nu \leq n$. Similar representations are obtained for ultraspherical and Jacobi polynomials.

G. Szegö (Stanford, Calif.).

Gatteschi, Luigi. Una proprietà degli estremi relativi dei polinomi di Jacobi. Boll. Un. Mat. Ital. (3) 8, 398–400 (1953).

Let $\varphi_{r,n}$ be the abscissa of the rth relative extremum of $P_n(\alpha, \beta; \cos\theta)$, where r is fixed. Then

$$\lim_{n\to\infty} (\sin \varphi_{r,n})^\alpha (\cos \varphi_{r,n})^\beta P_n(\alpha; \beta; \cos \varphi_{r,n}) = 2^{\alpha+\beta} J_\alpha(j),$$

where j is the rth zero of the Bessel function $J_{\alpha+1}(x)$. The proof is based on the "Hilb-Szegö formula" for Jacobi polynomials. *G. Szegö* (Stanford, Calif.).

Rimarenko, B. A. On a class of monotonic polynomials. Dopovidi Akad. Nauk Ukrain. RSR 1952, 96–99 (1952). (Ukrainian. Russian summary)

The author solves two extremal problems for the class $B_n^{(h)}$ of polynomials $y_n(x) = p_0x^n + p_1x^{n-1} + \cdots + p_n$ whose odd derivatives up to order $2h+1$ are non-negative on the whole real axis, and whose even derivatives up to order $2h$ are non-negative at -1. (1) Find the minimum oscillation L_n of $y_n(x)$ on $(-1, 1)$ if $p_0 = 1$, and p_1 and p_2 are assigned. (2) Find L_n if $y_n^{(2h+1)}(\xi)$ is assigned at a prescribed ξ.

R. P. Boas, Jr. (Evanston, Ill.).

Kalinovs'ka, S. S. On convergence of mean-power approximations to Čebyšev approximations for some interpolation processes in n-dimensional space. Dopovidi Akad. Nauk Ukrain. RSR 1952, 263–267 (1952). (Ukrainian. Russian summary)

The author generalizes to several variables results of Remez for the one-variable case [cf. Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. 1948, no. 10, 107–141 (1949); these Rev. 12, 93]. *R. P. Boas, Jr.* (Evanston, Ill).

Abhyankar, S. S. Note on positive polynomials. Amer. Math. Monthly 61, 184–187 (1954).

Let $P(x) = x^n + a_1x^{n-1} + \cdots + a_n$ be an arbitrary polynomial of degree n having only positive coefficients a_j. Let $m(n)$ be the smallest integer such that one may write $P(x) = P_1(x) + P_2(x) + \cdots + P_k(x)$, where $k \leq m(n)$ and where each $P_j(x)$ has only negative zeros. By induction, the author proves that $n/2 < m(n) \leq n/2 + 1$. *M. Marden.*

Stein, S. The fundamental theorem of algebra. Amer. Math. Monthly 61, 109 (1954).

A topological proof of the fundamental theorem of algebra, based on the fact that a map of a circle to itself of degree n ($n\neq 0$) may not be extended to a map of the interior to the circumference. (The second reference appears without relation.) *P. J. Hilton* (Cambridge).

Harmonic Functions, Potential Theory

Ozawa, Mitsuru. The topology of subharmonic functions. Kōdai Math. Sem. Rep. 1953, 97–116 (1953).

Let G be a plane region whose boundary B consists of ν continua, and let u be a (pseudo-) harmonic function in G. Morse and Heins have shown that the difference of the number of relative maxima of u on B and of the sum of the orders of the saddle points of u is $2-\nu$. The present result extends this theorem to the case of subharmonic functions provided certain regularity conditions on the function are met. In the subharmonic case one must modify the Morse-Heins result by the addition of the sum of the Euler numbers of the sets where u attains its minimum. *H. L. Royden.*

Arsove, Maynard G. Functions of potential type. Trans. Amer. Math. Soc. 75, 526–551 (1953).

Une fonction δ-sousharmonique entière est la différence $w=u-v$ de deux fonctions sousharmoniques dans tout le plan; on notera μ la mesure associée à w (i.e. la "distribution" $\mu=-\Delta w/2\pi$). On commence par adapter à ces fonctions les définitions et résultats fondamentaux de la théorie des fonctions méromorphes concernant l'ordre, l'exposant de convergence, le genre, etc. Les deux outils essentiels sont d'une part la fonction caractéristique $T_r(w)$ (valeur moyenne de sup (u, v) sur la circonférence $|z|=r$) déjà introduite par l'auteur [Trans. Amer. Math. Soc. 75, 327–365 (1953); ces Rev. 15, 526], d'autre part une représentation intégrale calquée sur la décomposition de Weierstrass en produits infinis, déjà étendue aux fonctions sousharmoniques de "genre fini" par W. R. Transue [Amer. J. Math. 65, 335–340 (1943); ces Rev. 4, 246] et, pour plus de deux dimensions, par M. Brelot [Ann. Inst. Fourier Grenoble 1, 121–156 (1950); ces Rev. 12, 258]. L'ordre de w est le nombre $\rho(w)=\lim\sup_{r\to\infty}(\log T_r(w)/\log r)$.

Plus originale est la partie consacrée aux fonctions "de type potentiel", i.e. les fonctions δ-sousharmoniques entières d'ordre nul dont la mesure associée a une variation totale finie ($\int|d\mu|<\infty$). Cette sous-classe ($\mathcal{P}$) contient les potentiels, les constantes, et aussi des fonctions n'ayant pas d'analogues dans la théorie des fonctions de variable complexe, car si $w=\log|f(z)|$, avec $f(z)$ méromorphe, est dans ($\mathcal{P}$), on a: $w=$potentiel+constante. La représentation à la Weierstrass donne ici:

$$(1)\quad w(z)=c-\int_{|\zeta|\leq 1}\log|z-\zeta|d\mu(\zeta)-\int_{|\zeta|>1}\log\left|1-\frac{z}{\zeta}\right|d\mu(\zeta),$$

où μ est la mesure associée et c une constante; inversement si μ est de variation totale finie, (1) a un sens et représente une fonction $w\in(\mathcal{P})$, et il y a correspondance biunivoque entre les $w\in(\mathcal{P})$ et les couples (c, μ). Des résultats de M. Heins [Ann. of Math. (2) 49, 200–213 (1948); ces Rev. 9, 341] conduisent au critère suivant: pour que w, δ-sousharmonique entière, soit dans ($\mathcal{P}$), il faut et il suffit que $T_r(w)=O(\log r)$ ($r\to\infty$).

On introduit la norme suivante sur ($\mathcal{P}$), qui est un espace vectoriel sur le corps des réels (et même un "vector lattice"): $w=|c|+\int|d\mu|$. Ainsi normé, ($\mathcal{P}$) est complet, et les potentiels sont denses dans ($\mathcal{P}$). Pour finir, on étudie diverses sortes de convergence dans ($\mathcal{P}$), et on donne des applications à la théorie du potentiel. *J. Deny* (Strasbourg).

Garabedian, P. R. Orthogonal harmonic polynomials. Pacific J. Math. 3, 585–603 (1953).

A complete orthonormal set of harmonic functions in oblate and prolate spheroids is determined for the case of the scalar products $(f, g)=\int\nabla f\cdot\nabla g$ and $[f, g]=\int fg$. Application is made to the extremum problem $[f_n, f_n]/(f, f)=\min$ within the class of harmonic functions. The kernel functions for both metrics are determined and the biharmonic Green's functions of a spheroid is expressed in terms of one of them. It is shown that one orthonormal set given is also orthogonal and complete under a scalar product based on surface integration with appropriate weight function. *M. Schiffer* (Stanford, Calif.).

Kametani, Shunzi. A note on a metric property of capacity. Nat. Sci. Rep. Ochanomizu Univ. 4, 51–54 (1953).

This note consists of a correction and several additions to the author's fundamental paper on modern potential theory [Jap. J. Math. 19, 217–257 (1945); these Rev. 7, 522]. These concern the important and well-known relation between capacity or transfinite diameter and Hausdorff measure which in its original form (for closed plane sets and logarithmic capacity) is due to Myrberg [Acta Math. 61, 39–79 (1933)], although it is implicit in the thesis of H. Cartan [Ann. Sci. Ecole Norm. Sup. (3) 45, 255–346 (1928)]. Theorem 13 (p. 234) of the author's 1945 paper states this relation for generalized capacities as follows (all sets lie in Euclidean space of two or more dimensions): "Let $h(r)$ be a Hausdorff measure function for which $\int_0^{\epsilon}h(r)d\Phi(r)$ is finite. Then, if E is a bounded Borel set of positive h-measure, the capacity of E with respect to $\Phi(r)$ is not zero." The proof, however, assumes that E contains a closed set of positive h-measure. This assumption, which is true if E has finite measure, is false (assuming the continuum hypothesis) for arbitrary measurable sets of infinite measure [Besicovitch, Acta Math. 62, 289–300 (1934); also Choquet, Bull. Soc. Math. France 74, 1–14 (1946); these Rev. 9, 419]. However, its truth for the case of $F_{\sigma\delta}$ sets and a special class of Hausdorff measures ($h(r)=r^s$ for some $s>0$) follows from a recent result of Besicovitch [Indagationes Math. 14, 339–344 (1952); these Rev. 14, 28] that any such set of infinite h-measure contains closed subsets whose h-measures take on all finite positive values. In the present note, therefore, the author proves the following alternative to the above result: Theorem 2: "Let $\Phi(r)=r^{-\alpha}$ for some $\alpha>0$, and let $h(r)$ be a Hausdorff measure function for which $\int_0^{\epsilon}h(r)d\Phi(r)$ is finite. Then, if E is an arbitrary set of outer capacity zero with respect to $\Phi(r)$, its h-measure is zero." Theorem 13 (for the case $\Phi(r)=r^{-\alpha}$) would follow from this if, for bounded Borel sets, capacity zero implies outer capacity zero. This follows, however, from recent results of Choquet on the "capacitability" of Borel and analytic sets [C. R. Acad. Sci. Paris 234, 35–37, 383–385, 784–786 (1952); these Rev. 13, 555, 633]. It should also be noted that Theorem 2 has been previously given by L. Carleson [thesis, Uppsala, 1950; these Rev. 11, 427], whose version also includes the logarithmic case. *B. Lepson* (Washington, D. C.).

Zin, Giovanni. Contributo alla risoluzione del problema piano di Dirichlet. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 14, 750–754 (1953).

Let D be a region bounded by a Jordan curve C whose parametrization $z(t)=x(t)+iy(t)$ is absolutely continuous. The author announces the result that it is possible to prove the existence in D of a harmonic function $u(x, y)$ which tends angularly to a given bounded integrable function $g(x, y)$ on C for almost all points of C. Various well-known properties of such a harmonic function are stated.

A. J. Lohwater (Ann Arbor, Mich.).

Zin, Giovanni. Risoluzione del problema piano di Dirichlet. Ann. Mat. Pura Appl. (4) 35, 203–254 (1953).

Proofs of the results announced in the paper of the preceding review. *A. J. Lohwater* (Ann Arbor, Mich.).

Rayner, M. E. A note on uniqueness proofs for boundary-value problems in potential theory and steady heat conduction. Quart. J. Mech. Appl. Math. 6, 385–390 (1953).

In the two-dimensional potential theory, the logarithmic singularities of the potential function at infinity in general preclude uniqueness theorems. For example, the function $\log r$ is harmonic outside the circle $r=1$, vanishes on this circle, but is not identically zero. The author discusses how uniqueness proofs can be obtained from the usual Green's identity in the case of a semi-infinite region bounded by a curve which goes to infinity in two directions. The potentials discussed are assumed to satisfy a regularity condition at infinity; namely to be of the form $A \log r+f(\theta)+g(r, \theta)$, where g is regular at infinity. All three ordinary boundary-value problems are discussed and interpretations given in terms of image systems of charges. *J. W. Green.*

Huth, J. H. Mixed boundary value problems in potential theory. J. Franklin Inst. 257, 121–124 (1954).

The author considers the solution φ of Laplace's equation in the unit circle with prescribed boundary values in the upper half of the circle and prescribed normal derivative in the lower half. The singularities that occur at the juncture points are interpreted in terms of dipoles.

M. H. Protter (Berkeley, Calif.).

Differential Equations

Have

★**Sikkema, P. C. Differential operators and differential equations of infinite order with constant coefficients. Researches in connection with integral functions of finite order.** P. Noordhoff N. V., Groningen-Djakarta, 1953. 223 pp. 11.50 florins; cloth 13.50 florins.

The author considers differential operators

$$F(D)=\sum_{n=0}^{\infty} a_n D^n, \quad D=d/dz,$$

and discusses two topics: the applicability of $F(D)$ to a given class of entire functions (and the characteristics of the functions which result from applying it), and the existence of solutions of the equation $F(D)y=h(z)$, where $h(z)$ is an entire function of a specified class. He goes into much greater detail than others who have dealt with these problems and obtains more precise results. In particular, he deals with cases in which the function $F(z)$ is not necessarily analytic at 0; he not only determines upper bounds for the growth of $F(D)y$, but also lower bounds; and he shows not only that $F(D)y=h$ has solutions of at most a certain rate of growth, but that it has solutions of precisely a certain rate of growth. The difference equation $y(z+\omega)-y(z)=h(z)$ is included as a special case. For example, let $y(z)$ be a transcendental entire function of growth not exceeding order 1, minimum type; let $F(D)$ be an operator (not identically 0) applicable to all entire functions which are not of higher order of type than $y(z)$; then $y(z)$ is transformed into a function $h(z)$ of the same order and type. Conversely, if $h(z)$ has growth not exceeding order 1, minimum type, the equation $F(D)y=h$ has a solution of the same order and type, and none of lower order or type.

Some of the author's results generalize part of J. M. Whittaker's work on asymptotic periods: instead of considering numbers ω such that $(e^{\omega D}-1)y$ is of slower growth than y, he considers conditions under which a more general transform $F(D)y$ is of slower growth than y. [For the operator $e^{\omega D}-1$ this had already been done by S. Scott, Proc. Cambridge Philos. Soc. 31, 543–554 (1935).]

The author's principal methods are the rearrangement of power series and the use of infinite systems of linear equations. *R. P. Boas, Jr.* (Evanston, Ill.).

Szarski, J. Sur les systèmes d'inégalités différentielles ordinaires remplies en dehors de certains ensembles. Ann. Soc. Polon. Math. 24 (1951), no. 2, 1–8 (1954).

The inequalities considered have the form $D^+y^i \leqq f^i(x, y^1, \cdots, y^n)$, where D^+ denotes the upper right derivate with respect to x. Let z^i be an upper integral of $z^{i\prime}= f^i(x, z', \cdots, z^n)$. The paper gives a sufficient condition that $y^i \leqq z^i$. *J. M. Thomas* (Durham, N. C.).

Hukuhara, Masuo. Sur une généralisation d'un théorème de Kneser. Proc. Japan Acad. 29, 154–155 (1953).

The author considers a differential inequality

$$(1) \qquad |y'-f(x, y)| \leqq F(x, y),$$

with x and F real and y and f real vectors, on a point set S: $0 \leqq x \leqq 1$, $|y|<\infty$. It is stated that if A is a continuum in S then the family of all those solutions y of (1) which intersect A is a continuum in the space C of continuous functions. The topology of C is not specified. *F. A. Ficken.*

Franchini, Lucia. Criteri d'unicità per gli integrali di un sistema di equazioni differenziali ordinarie. Ann. Univ. Ferrara. Sez. VII. (N.S.) 2, 53–69 (1953).

This paper gives uniqueness criteria for a system of n ordinary differential equations of the first order. For $n=1$ they generalize those of Cafiero [Giorn. Mat. Battaglini (4) 2(78), 10–41 (1948); these Rev. 10, 457], Zwirner [Rend. Sem. Mat. Univ. Padova 11, 90–96 (1940); these Rev. 8, 206] and others. *J. M. Thomas* (Durham, N. C.).

Hayashi, Kyuzo. On the solutions of the system of ordinary differential equations. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 28, 19–25 (1953).

The system is $y'=f(x, y)$, where y is an unknown vector, f is measurable in x and continuous in y, $|f| \leqq M(x)$ and M is Lebesgue integrable. The paper removes the continuity assumption from the theorems of Kneser [S.-B. Preuss. Akad. Wiss. 1923, 171–174] and Fukuhara [Jap. J. Math. 5, 345–350 (1929)].

J. M. Thomas (Durham, N. C.).

Froda, Alexandru. Equations différentielles Lavrentieff et les fonctions Pompeiu. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 4, 801–814 (1952). (Romanian. Russian and French summaries)

Let $g(x)$ increase on an interval. Let its derivative $g'(x)$ have a root on each subinterval. Let h be the inverse of g. Through each point of a rectangle there pass infinitely many solutions $y(x)$ of $h'(y)y'(x)=1$. The effect of a change of variables on the solutions is studied. There is given a change which, in general, converts the solutions at each point into a pencil of straight lines with finite center.

J. M. Thomas (Durham, N. C.).

Yoshizawa, Taro. On the evaluation of the derivatives of solutions of $y''=f(x,y,y')$. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 28, 27–32 (1953).

Let (x,y) be in a bounded closed set and y' in $-\infty<y'<\infty$. Let y satisfy $y''=f(x,y,y')$, where f is continuous. The paper gives a rather complicated necessary and sufficient condition that $x_0\leqq x$, $|y'(x_0)|\leqq a$ imply $|y'(x)|<b(a)$ for arbitrary a and suitable b. *J. M. Thomas.*

Gianuizzi, Maria. Un criterio di esistenza per un problema al contorno relativo all'equazione

$$y^{(\mathrm{iv})}=\lambda f(x,y,y',y'',y''').$$

Ann. Univ. Ferrara. Sez. VII (N.S.) 2, 35–43 (1953).

Sufficient conditions are given in order that there exist at least one solution $(y(x),\lambda)$ of the equation

$$y^{(\mathrm{iv})}(x)=\lambda f(x,y(x),y'(x),y''(x),y'''(x))$$

such that, with x_k and y_k fixed $(k=1,\cdots,5)$, $y(x_k)=y_k$.

F. A. Ficken (Knoxville, Tenn.).

Saito, Tosiya. Sur les solutions autour d'un point singulier fixe des équations différentielles du premier ordre. Kōdai Math. Sem. Rep. 1953, 121–126 (1953).

The equation (E) $Q(x,y)y'=yP(x,y)$ is studied in the neighborhood of an isolated fixed singularity at $x=0$. It is assumed that $P=\sum_{k=0}^{\infty}a_k(x)y^k$ and $Q=\sum_{k=0}^{\infty}b_k(x)y^k$ where a_k and b_k are analytic and single-valued, while, for each x, Q and yP are relatively prime. Let x_0 be a regular point of (E) and let $\varphi(x;x_0,y_0)$ satisfy (E) and have $\varphi(x_0;x_0,y_0)=y_0$. Let A denote an annular region centered at $x=0$ and containing no singular point. If $|y_0|$ is sufficiently small and the residue r of $a_0(x)/b_0(x)$ at $x=0$ is irrational, then in A each branch of $\varphi(x;x_0,y_0)$ is expressible in the form $\sum_{k=1}^{\infty}x^{rk}v_k(x)$ where each $v_k(x)$ is analytic and single-valued in A. The functions $v_k(x)$ can be obtained by quadratures.

F. A. Ficken (Knoxville, Tenn.).

Krasovskii, N. N. On stability of solutions of a system of two differential equations. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 651–672 (1953). (Russian)

Consider the system

$$\dot{x}_1=h_{11}x_1+h_{12}x_2,\quad \dot{x}_2=h_{21}x_1+h_{22}x_2.$$

If the h_{ij} are constants the system will be asymptotically stable in the large provided that

$$h_{11}+h_{22}<0,\quad h_{11}h_{22}-h_{21}h_{12}>0.$$

The problem discussed is in what measure this still holds when the h_{ij} depend on the x_i. Erugin [same journal 14, 459–512, 659–664 (1950); 16, 620–628 (1952); these Rev. 12, 412; 14, 376] and Malkin [ibid. 16, 365–368 (1952); these Rev. 14, 48] discussed a system with just one non-linear element. The author discusses at length the system

$$\dot{x}=f_1(x)+ay,\quad \dot{y}=f_2(x)+by,$$

the same with $f_2(y)$ instead of $f_2(x)$, the same with $f_1(y)$ and $f_2(x)$ and

$$\dot{x}=f_1(x)+f_2(y),\quad \dot{y}=ax+by.$$

In each case a number of sufficient conditions for asymptotic stability in the large are given but they are too complicated to be indicated here. *S. Lefschetz* (Princeton, N. J.).

Vinograd, R. È. Negative solution of a question on stability of characteristic exponents of regular systems. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 645–650 (1953). (Russian)

The author shows that the characteristic exponents of the solutions of the nonlinear system $z'=A(t)z+F(t,z)$ are not equivalent to those of the linear system $z'=A(t)z$ for unrestricted variable matrices $A(t)$, even under strong assumptions concerning the order of smallness of the nonlinear term $F(t,z)$. *R. Bellman* (Santa Monica, Calif.).

Burdina, V. I. On boundedness of the solutions of a system of differential equations. Doklady Akad. Nauk SSSR (N.S.) 93, 603–606 (1953). (Russian)

The author gives some conditions of rather complicated type which ensure the boundedness of solutions of the equation $y''+p(t)y=0$, where $p(t)$ is a periodic function of t.

R. Bellman (Santa Monica, Calif.).

Daleckiĭ, Yu. L. On the asymptotic solution of a vector differential equation. Doklady Akad. Nauk SSSR (N.S.) 92, 881–884 (1953). (Russian)

The equation discussed is

$$A(\tau,\epsilon)\frac{dq(t)}{dt}=iB(\tau,\epsilon)q(t)+p(\tau,\epsilon)e^{i\theta(t,\epsilon)},$$

where $\tau=\epsilon t$, p and q are vectors in Hilbert Space, and A, B are linear operators in that space. The results do not admit of a short resumé. *J. L. B. Cooper* (Cardiff).

Kruming, A. A. Estimate of the radius of convergence of power series in a small parameter which represent periodic solutions of systems of differential equations. Ukrain. Mat. Žurnal 5, 434–438 (1953). (Russian)

Consider an analytical vector system $\dot{X}=F(X;\lambda)$, analytic also in λ and let it possess for $\lambda=0$ an isolated periodic solution $\varphi(t)$ with period 2π. If $X=x+\varphi(t)$ it is assumed that one may expand $F(x+\varphi(t);\lambda)$ in power series of the x_i and of λ for $\|x\|<r$, $|\lambda|<r_0$, the coefficients being then periodic and of period 2π. Under certain conditions the system has a unique periodic solution $X(t;\lambda)\to\varphi(t)$ as $\lambda\to 0$, for λ small (Poincaré). Using a Perron estimate [Math. Ann. 113, 292–303 (1936), p. 300] and many majorants the author calculates an explicit range for the radius of convergency of the known power series periodic solution $X(t;\lambda)$. *S. Lefschetz* (Princeton, N. J.).

Stoppelli, Francesco. Su un'equazione differenziale della meccanica dei fili. Rend. Accad. Sci. Fis. Mat. Napoli (4) 19 (1952), 109–114 (1953).

Let $p(t)>0$, $q(t)$, $f(t)$ be continuous functions with the period T. Then it is proved that the differential equation

$$y''+y'|y'|+q(t)y'+y-p^2(t)y^2=f(t)$$

possesses at least one solution with period T. For the proof the differential equation is converted into a non-linear

integral equation whose kernel is Green's function for the operator $y''+y$ with periodicity as boundary condition. The integral equation is then shown to possess a solution by using a fixed-point theorem in a Banach space. The author states that the same technique applies to the more general differential equation

$$y''+q_0(t)y'|y'|+q(t)y'+p_0(t)y-p^2(t)y^3=f(t)$$

provided $|q_0(t)|$ is always positive. *W. Wasow.*

Modona, Lionella Neppi. Su di una equazione differenziale non lineare del secondo ordine. Boll. Un. Mat. Ital. (3) **8**, 428–441 (1953).

In the differential equation

$$y''+y'|y'|+y'q(t)+y-p^2y^3=0$$

it is assumed that $q(y)$ is positive and differentiable and that $q'(y)=O(|y|^3)$, as $y\to\infty$. The author determines a region Z of the (y, y')-plane such that any integral curve whose initial values for $t=0$ lie in Z remains in Z for $t>0$ and satisfies the stability condition $\lim_{t\to\infty}[y^2(t)+y'^2(t)]=0$. For an integral characterized by initial values outside Z it is proved, conversely, that $\lim_{t\to T}[y^2(t)+y'^2(t)]=\infty$ for some finite T. The region Z is bounded by two integral curves that pass, respectively, through the singular points $(-1/p, 0)$ and $(1/p, 0)$ in the (y, y')-plane. *W. Wasow.*

***Oppelt, Winfried. Theorie der Regelung und Steuerung.** Naturforschung und Medizin in Deutschland, 1939–1946, Band 4. Angewandte Mathematik, Teil II, pp. 127–135. Verlag Chemie, Weinheim, 1953. DM 10.00.

Fettis, Henry E. On a differential equation occurring in the theory of heat flow in boundary layers with Hartree's velocity profiles. J. Aeronaut. Sci. **21**, 132–133 (1954).

In order to find the solution of the differential equation

$$d^2\theta/d\xi^2+\tfrac{1}{2}\xi^2d\theta/d\xi-\epsilon\xi\theta=0$$

subject to the condition $\theta(0)=1$, $\theta(\infty)=0$, the variables are changed to $t=\xi^3/6$, $\phi=t^{1/6}e^{t/2}\theta$, which reduces the differential equation to Whittaker's confluent hypergeometric equation. This leads to an explicit expression for the desired solution as a definite integral, which can be used to derive an exact formula for the heat transfer coefficient.

W. Wasow (Los Angeles, Calif.).

Schaffner, J. S. Almost sinusoidal oscillations in nonlinear systems. III. Transient phenomena. University of Illinois Bulletin. Engineering Experiment Station Bulletin Series no. **421**, 39 pp. (1953).

Taam, Choy-tak. On the complex zeros of functions of Sturm-Liouville type. Pacific J. Math. **3**, 837–843 (1953).

Two theorems are proved giving upper bounds to the number of zeros of a solution of $W''+Q(z)W=0$ in a domain in which $Q(z)$ is analytic. *G. E. H. Reuter.*

Taam, Choy-Tak. On the solutions of second order linear differential equations. Proc. Amer. Math. Soc. **4**, 876–879 (1953).

A criterion ensuring that no solution of

$$(P(x)W')'+Q(x)W=0$$

(with complex P and Q) vanishes more than once on an interval I is obtained, extending an earlier result for $P(x)\equiv 1$ [Portugaliae Math. **12**, 57–72 (1953); these Rev. **14**, 873]. Two applications of the criterion are made.

G. E. H. Reuter (Manchester).

Ledermann, W., and Reuter, G. E. H. Spectral theory for the differential equations of simple birth and death processes. Philos. Trans. Roy. Soc. London. Ser. A. **246**, 321–369 (1954).

The differential system
$dp(t)/dt=p(t)A$, $p(t)=(p_{ij}(t))$, $p(0)=(\delta_{ij})$, $i, j=0, 1, \cdots$, where

$$A=(a_{ij})=\begin{bmatrix} -\lambda_0 & \lambda_0 & & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & & \\ & & \cdots & \cdots & \cdots & \\ & & & \mu_n & -(\lambda_n+\mu_n) & \lambda_n \\ & & & & \cdots & \cdots \end{bmatrix}$$

is solved by approximating A by

$$A^{(n)}=(a_{ij}^{(n)}),\quad a_{ij}^{(n)}=\begin{cases} a_{ij} & \text{when } i \text{ and } j\leq n+1,\\ 0 & \text{otherwise,}\end{cases}$$

and letting $n\to\infty$ in the corresponding solution

$$\exp(tA^{(n)})=\sum_{r=0}^{n}\exp(t\alpha_r^{(n)})D_r^{(n)}.$$

Here $\alpha_r^{(n)}$ are the eigenvalues of $A^{(n)}$ and the $D_r^{(n)}$ the corresponding idempotent matrices. Another approximation is also considered by replacing $a^{(n)}_{n+1,n+1}$ in $A^{(n)}$ by $-\mu_n$. The results are applied to the special case when

$$\lambda_{n+1}/\lambda_n=1+an^{-1}+O(n^{-2}),$$
$$\lambda_n/\mu_n=c(1+bn^{-1}+O(n^{-2})),\quad c>0.$$

It is shown, that for certain values of the parameters a, b and c, the two approximations yield different limits.

K. Yosida (Osaka).

Pöschl, Th. Über Hauptschwingungen mit endlichen Schwingweiten. II. Ing.-Arch. **21**, 396–398 (1953).

This is a further development of ideas which the author has presented in a previous note [Ing.-Arch. **20**, 189–194 (1952); these Rev. **14**, 324]. It is now shown that the differential equation defining what the author calls the Grenzkurven of the vibrations is the Hilbert transversality condition for a certain variational problem. *L. A. MacColl.*

Drâmbă, C. Sur la distribution des trajectoires autour d'un point singulier isolé. Acad. Repub. Pop. Române. Bul. Şti. Sect. Şti. Mat. Fiz. **3**, 333–340 (1951). (Romanian. Russian and French summaries)

This note gives results about the number of tangents to integral curves at an isolated singular point of the system $dx_1:\cdots:dx_n=X_1:\cdots:X_n$, particularly in the case where the characteristic roots of $\|\partial X_i/\partial x_j\|$ are real.

J. M. Thomas (Durham, N. C.).

Germay, R. H. Sur l'application de la méthode des approximations successives à la détermination de l'intégrale de Darboux et de sa généralisation dans le cas d'une équation aux dérivées partielles du premier ordre de forme résolue. I, II. Bull. Soc. Roy. Sci. Liège **22**, 359–367, 423–436 (1953).

The method of successive approximations is applied to find Darboux's integral of the equation

$$p_1=f(x_1, \cdots, x_n, z, p_2, \cdots, p_n).$$

Analytic functions of the usual parameters are employed.

J. M. Thomas (Durham, N. C.).

Germay, R. H. Sur des systèmes d'équations récurrentes aux dérivées partielles du second ordre. Bull. Soc. Roy. Sci. Liège 22, 504–513 (1953).

The derivatives $\partial^2 z_{ij}/\partial x \partial y$ of the unknowns

$$z_{ij}\ (i=1, \cdots, k;\ j=1, 2, \cdots, \infty)$$

are given as functions F_{ij} of the independent variables x, y, the unknowns z_{lm} and their first derivatives ($l=1, \cdots, k$; $m=j, j+1, \cdots, j+p-1$, where p is fixed). The functions F_{ij} are bounded and satisfy a Lipschitz condition. The initial values are bounded and satisfy additional inequalities. Existence and uniqueness are proved. *J. M. Thomas.*

Torcoli, Emilia. Su la determinazione delle soluzioni omogenee di equazioni alle derivate parziali. Rivista Mat. Univ. Parma 4, 213–218 (1953).

The substitution $z=x^a f(t)$, $y=xt$ converts a partial differential equation P in unknown z into an ordinary equation E in unknown f and parameters a,x. If P is to have a solution which is homogeneous of degree a in x, y, equation E must have a solution homogeneous of degree 0 in x. Examples are given to illustrate the cases in which E has general solution yielding a homogeneous solution of P: (i) for arbitrary a; (ii) for particular a; and (iii) for no a. *J. M. Thomas.*

Szarski, J. Sur un système d'équations aux dérivées partielles du premier ordre complètement intégrable. Ann. Soc. Polon. Math. 24 (1951), no. 2, 9–16 (1954).

The system treated is of the Koenig type and passive [Riquier, Les systèmes d'équations aux dérivées partielles, Gauthier-Villars, Paris, 1910, p. xv]. In a region defined by specific inequalities the present paper proves the existence of a unique solution of class C'' on the assumption that system and initial determination are of class C'''.

J. M. Thomas (Durham, N. C.).

Hornich, Hans. Häufigkeit von regulären Lösungen bei gewissen partiellen Differentialgleichungen erster Ordnung. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 11, 125–133 (1952).

Soit l'équation aux dérivées partielles

$$\sum_{i=1}^{n} \alpha_i x_i (\partial u/\partial x_i) = f(x_1, \cdots, x_n),$$

où $\alpha_1, \cdots, \alpha_n$ désignent des constantes et f une série de puissances de $x_1, \cdots, x_n$, convergente au voisinage de l'origine. L'auteur donne les conditions nécessaires et suffisantes pour que l'équation étudiée admette également pour solution u une série de puissances de $x_1, \cdots, x_n$ convergente au voisinage de l'origine et étudie l'ensemble des points $(\alpha_1, \cdots, \alpha_n)$ pour lesquels u converge quelle que soit la série f ou pour lesquels u ne converge que pour des formes particulières de la série f. *H. G. Garnir* (Liège).

Hornich, Hans. Risolubilità di generali equazioni lineari a derivate parziali mediante serie di potenze. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 15, 7–10 (1953).

The conditions for the existence of power series solutions previously given [Rend. Circ. Mat. Palermo (2) 2, 46–52 (1953); these Rev. 15, 429] are now extended to Laurent series. *J. M. Thomas* (Durham, N. C.).

Malgrange, Bernard. Equations aux dérivées partielles à coefficients constants. I. Solution élémentaire. C. R. Acad. Sci. Paris 237, 1620–1622 (1953).

Malgrange, Bernard. Equations aux dérivées partielles à coefficients constants. II. Equations avec second membre. C. R. Acad. Sci. Paris 238, 196–198 (1954).

These notes announce theorems on linear partial differential equations $Lf=g$ with constant coefficients, in the context of L. Schwartz' theory of distributions. In the first note g is a continuous linear form on an appropriate function space and the Fourier transform, an inequality from complex analysis, and the Hahn-Banach theorem are combined to yield a solution f in the same space; if g is the "δ-function" then f is an "elementary solution". In the second note g may be in the space of infinitely differentiable functions, or in the space of finite sums of derivatives of continuous functions, or (and here L is assumed elliptic) in the space of distributions. It is shown again that a solution f can be found in the space in which g is assumed to lie.

F. A. Ficken (Knoxville, Tenn.).

Lions, Jacques-Louis. Problèmes aux limites et conditions à l'infini. C. R. Acad. Sci. Paris 237, 1617–1620 (1953).

This note states three theorems (and a fourth, ignored here) for unbounded regions similar to those stated earlier for bounded regions [same C. R. 236, 2373–2375, 2470–2472; 237, 12–14 (1953); these Rev. 15, 317]. A linear differential operator L is elliptic in one theorem and parabolic in the other two. Each theorem states that L maps a certain linear space isomorphically onto another and gives information about the inverse operator. Behavior at ∞ is controlled by use of exponential factors. *F. A. Ficken.*

Tashiro, Shizuko, and Ono, Akira. On the regularity of distributions defined by some differential equations. Mem. Fac. Sci. Kyūsyū Univ. A. 8, 93–107 (1953).

L. Schwartz [Théorie des distributions, Hermann, Paris, 1951, t. II, ch. VI, §6, pp. 36–48 et spécialement p. 46; ces Rev. 12, 833] a montré que si, dans R_n, $\Delta T \in L^p$ ($p \geq 1$) sur tout compact, $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$, alors, la distribution $T \in L^q$ sur tout compact, pour $q=q(p, n)$, ou satisfait à une condition de Lipschitz d'ordre s ($0 \leq s \leq 1$) dans L^q sur tout compact, pour $q=q(p, n, s)$. Les auteurs donnent une démonstration détaillée de cette proposition. Ils énoncent et démontrent des propriétés analogues, Δ étant remplacé par ∇^k, $k=\frac{1}{2}n$ (n pair), $\frac{1}{2}(n-1)$, $\frac{1}{2}(n+1)$ (n impair), $\nabla \equiv \partial^2/\partial x_n^2 - \partial^2/\partial x_1^2 - \cdots - \partial^2/\partial x_{n-1}^2$, ou encore par $D \equiv \partial/\partial x_n - \partial^2/\partial x_1^2 - \cdots - \partial^2/\partial x_{n-1}^2$. *H. G. Garnir.*

Brousse, Pierre. Quelques propriétés des intégrales d'une classe d'équations singulières dans certains domaines. C. R. Acad. Sci. Paris 237, 1381–1383 (1953).

Continuing the work of a previous paper [same C. R. 236, 1731–1732 (1953); these Rev. 14, 984], the author obtains an integral representation for a certain type of domain of the solution $S(x, y)$ of the equation

$$(*)\qquad S_{xx}+S_{yy}+(k+2)y^{-1}S_y=0,$$

where $k>0$ is a constant. From this representation it can be deduced that (i) any solution of (*) regular and positive for $y>0$ and bounded on the x-axis is a constant, and (ii) any solution regular for $y>0$, bounded on the x-axis and vanishing uniformly at infinity is identically zero. The Dirichlet problem is discussed for (*) and the equation (**) $V_{xx}+V_{yy}-ky^{-1}V_y=0$ for certain domains whose boundary consists of two half-infinite line segments and an

arc connecting them. Series developments for $V(x, y)$ and the Green's function for (**) in a strip are given in terms of a set of fundamental functions. *M. H. Protter.*

Garabedian, P. R., and Schiffer, M. Convexity of domain functionals. J. Analyse Math. 2, 281–368 (1953).

In this paper the authors generalize their previous work on the method of interior variations to develop a rigorous theory of variation of domain functions in space as well as in the plane, and from the second-variation expressions derived for the capacity, virtual mass, etc., they deduce several convexity theorems for these domain functionals. They define interior variations of a three-dimensional domain D by means of differentiable mappings of D depending on a small parameter ϵ. By referring all varied quantities back to the original domain D through the infinitesimal mappings, the authors reduce the study of the domain dependence of the domain functions associated with a given linear differential equation to an investigation in the fixed domain D of the dependence of the solution on the coefficients of the equation; this investigation is then carried through by known methods based on the theory of integral equations. The resulting variational formulas do not depend on strong smoothness assumptions on the boundary of the domain, but in the case of sufficiently smooth boundaries a by-product is the rigorous derivation of the classical Hadamard variational formula. The second variations of the domain functions, obtained from the expansion of these functions in powers of ϵ, yield a variety of convexity theorems depending on the manner of domain variation. For example, if the boundary is shifted along level surfaces of a harmonic function U, the capacity of the domain with respect to a fixed interior point turns out to be a convex function of U. Among other results are convexity theorems for the eigenvalues of the vibrating membrane and for the virtual mass, certain uniqueness theorems, and the proof of existence of vortex sheets in axially symmetric potential flow. *D. Gilbarg* (Stanford, Calif.).

Hadamard, J. Equations du type parabolique dépourvues de solutions. J. Rational Mech. Anal. 3, 3–12 (1954).

L'autore dimostra che esistono funzioni continue $f(x, y)$ tali che l'equazione $\partial^2u/\partial x^2-\partial u/\partial y=f(x, y)$ sia sprovvista di soluzioni. Per soluzione de tale equazione deve intendersi una funzione che la verifichi ovunque in un assegnato dominio. [Per un risultato analogo relativo alle equazioni ellittiche vedi A. Wintner, Amer. J. Math. 72, 731–738 (1950); questi Rev. 12, 704.] *C. Miranda* (Napoli).

Browder, Felix E. Errata: Linear parabolic differential equations of arbitrary order; general boundary-value problems for elliptic equations. Proc. Nat. Acad. Sci. U. S. A. 39, 1298 (1953).

See same vol., 185–190 (1953); these Rev. 14, 984.

Birkhoff, Garrett, and Kotik, Jack. Note on the heat equation. Proc. Amer. Math. Soc. 5, 162–167 (1954).

Uniqueness and existence theorems for the one-dimensional heat conduction equation are discussed in terms of the heat content, rather than the temperature, of the conductor. If $H(x)$ is of bounded variation in every finite interval, $H(0)=0$ and $|H(x)|<M\exp[cx^2]$, then there exists a solution of the heat equation $u_{xx}=u_t$ on $0<t<\frac{1}{4}c$, $-\infty<x<\infty$ such that if $\int_0^x u(y,t)dy=H(x,t)$, then $H(x,t)\to H(x)$ as $t\to 0+$. If $|H(x,t)|<K\exp[Nx^2]$ for $0<t<T$ and $H(x,t)\to 0$ as $t\to 0+$ for almost all x, then $u(x,t)=0$ on $0<t<T$. *J. L. B. Cooper* (Cardiff).

Paterson, S. Conduction of heat from local sources in a medium generating or absorbing heat. Proc. Glasgow Math. Assoc. 1, 164–169 (1953).

A semi-infinite medium at temperature zero is put into contact over the face $x=0$ with a well stirred fluid at temperature T_0. The temperature of the fluid is assumed equal at all times to the surface temperature of the medium in which heat is generated or absorbed at a rate proportional to the temperature. After suitable changes of the variables this leads to consideration of the following problem:

$$\partial u/\partial\tau=\partial^2u/\partial x^2+Hu;\quad u=0,\ \tau=0;\quad u=u',\ x=0;$$
$$w(\partial u/\partial x)=du'/d\tau,\ x=0;\quad u'=1,\ \tau=0.$$

This problem is solved for both the linear and spherical cases by the use of the Laplace transform method. Applications are given illustrating both cases. *C. G. Maple.*

Kim, E. I. The propagation of heat in two dimensions in an infinite inhomogeneous body. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 555–568 (1953). (Russian)

The author considers the problem of the propagation of heat in two dimensions in two plates joined along a straight line. Taking the y-axis parallel to this line, the mathematical formulation of the problem leads to the equations

$$(1)\quad \frac{\partial u}{\partial t}=a_1^2\Delta u \text{ for } x<x_0,\quad \frac{\partial u}{\partial t}=a_2^2\Delta u \text{ for } x>x_0$$
$$\left(\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

subject to the initial conditions

$$(2)\qquad u(x, y, 0)=f(x, y),\quad -\infty<x, y<\infty,$$

and the boundary conditions,

$$(3)\qquad \begin{aligned} u(x_0-0, y, t)&=u(x_0+0, y, t),\\ k_1u_x(x_0-0, y, t)&=k_2u_x(x_0+0, y, t).\end{aligned}$$

The problem is reduced to the consideration of two integral equations through the use of Green's functions. These are reduced to a single integral equation which is solved in series form. When the initial function $f(x, y)$ does not depend upon y the solution may be expressed in terms of improper integrals. *C. G. Maple* (Ames, Iowa).

Alekseeva, O. P. A closed solution of certain boundary problems of mathematical physics. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 627–630 (1953). (Russian)

Three problems are considered, the first seeks the temperature distribution in a slab with zero initial temperature and constant but different temperatures along the two edges. The second problem is concerned with finding a function V which is related to the solution of the temperature distribution $v(x, t)$ in a slab through the equation

$$\frac{dV}{dt}=g+\frac{2\rho\nu}{\sigma}\left(\frac{\partial v}{\partial x}\right)_{x=0}$$

and through the conditions $v=V$ for $x=0$, $v=V=0$ for $x=0$, $t=0$. The third problem is the same as the first, with the exception that the constant temperature along one edge is replaced by an arbitrary function of time.

The object of the author is to obtain closed solutions for these problems. The method used is the classical Laplace transform technique. It is pointed out that solutions may

be obtained in terms of Fourier series or in terms of error functions. A third method expresses the solutions in terms of improper integrals. Mention is made of the fact the same method may be applied to the problem of the temperature distribution in a slab subject to arbitrary initial and boundary conditions. *C. G. Maple* (Ames, Iowa).

MacColl, L. A. Geometrical properties of two-dimensional wave motion. Amer. Math. Monthly **61**, 96–103 (1954).

L'auteur étudie, dans un domaine ouvert connexe donné, les solutions de l'équation $u_{xx}+u_{yy}=c^{-2}u_{tt}$ $(c>0)$, qui possèdent la forme $u=\exp[\alpha(x,y)+i\beta(x,y)+i\omega t]$ $(\omega>0)$. Il exprime les conditions auxquelles doivent alors satisfaire les courbes $\alpha(x,y)=\text{const.}$ (courbes d'amplitude constante) ou $\beta(x,y)=\text{const.}$ (courbes de phase constante). Ces conditions sont relativement compliquées. Cependant l'auteur signale différents cas particuliers intéressants où ces courbes peuvent être ou sont seulement des droites parallèles ou des circonférences concentriques. *H. G. Garnir* (Liège).

Conti, Roberto. Sul problema di Darboux per l'equazione $z_{xy}=f(x,y,z,z_x,z_y)$. Ann. Univ. Ferrara. Sez. VII. (N.S.) **2**, 129–140 (1953).

The Darboux problem is the characteristic initial value problem: $z_{xy}=f(x,y,z,z_x,z_y)$, $z(x,0)=\sigma(x)$ $(0\leq x\leq a)$, and $z(0,y)=\tau(y)$ $(0\leq y\leq b)$. A Lipschitzian solution $z(x,y)$ is here obtained for the familiar equivalent integral equation under the assumptions that σ and τ are Lipschitzian, while f is continuous, bounded, and uniformly Lipschitzian in z_x and z_y. The proof applies a two-dimensional version of an approximation process used by Tonelli for ordinary differential equations. *F. A. Ficken* (Knoxville, Tenn.).

De Giorgi, Ennio. Osservazioni relative ai teoremi di unicità per le equazioni differenziali a derivate parziali di tipo ellittico, con condizioni al contorno di tipo misto. Ricerche Mat. **2** (1953), 183–191 (1954).

The subject matter of this paper is of interest in connection with the uniqueness of solutions of boundary-value problems of mixed type for elliptic equations [see Fichera, Ann. Scuola Norm. Super. Pisa (3) **1**, 75–100 (1949); these Rev. **11**, 724]. The author constructs a continuous function u in the half disc D: $x^2+y^2\leq 1$, $y\geq 0$, which is harmonic in the interior, has finite Dirichlet integral, and satisfies the additional conditions: (i) $u=0$ on the curved boundary arc of D; (ii) there exists a set N on the straight boundary segment D_1 of D, having linear measure zero, such that u_x, u_y are continuous on $D-N$ and such that $u_y=0$ on D_1-N. *L. Nirenberg* (New York, N. Y.).

Protter, M. H. An existence theorem for the generalized Tricomi problem. Duke Math. J. **21**, 1–7 (1954).

L'auteur démontre dans cet article un résultat d'une extrême importance: le théorème d'existence pour un problème de Tricomi généralisé pour une équation autre que celle de Tricomi. L'équation à laquelle s'applique l'analyse est de la forme $K(y)u_{xx}+u_{yy}=0$, $K(y)$ étant une fonction croissante possédant une dérivée troisième continue et telle que $K(0)=0$, $K'(0)>0$. Les seuls cas étudiés jusqu'alors étaient $K(y)=y$ (Tricomi), $K(y)=y^\alpha$, $\alpha>0$ (Gellerstedt).

Il est difficile de donner une analyse d'un papier excessivement concis; disons cependant que l'auteur utilisant une transformation connue met l'équation donnée sous la forme $yu_{xx}+u_{yy}-C(y)u=0$, en faisant ainsi apparaitre les termes de l'équation de Tricomi ce qui lui permettra d'utiliser habilement la méthode de ses prédécesseurs. Cette méthode consiste à écrire les équations intégrales aux-quelles doivent satisfaire les valeurs de la fonction inconnue et de sa dérivée normale le long de la ligne parabolique et à prouver que ces équations ont une solution unique. L'auteur prouve essentiellement que, dans le cas général qu'il considère, ces équations ont les mêmes singularités et que l'analyse faite par ses prédécesseurs permet de déduire la même conclusion. Peut-être regrettera-t-on que pour un résultat d'une telle importance l'auteur ait préféré esquisser les grandes lignes de la démonstration plutot que de fournir une démonstration complète. De continuelles références sont faites aux précédents travaux, ce qui rend l'étude précise de ce mémoire assez difficile. Sans doute resterait-il encore à se débarrasser de quelques restrictions relatives à la partie du contour située dans le domaine elliptique, dues vraisemblablement à la méthode de démonstration qui sépare brutalement le domaine où est étudié le problème en deux parties elliptique et hyperbolique bien distinctes. *P. Germain.*

Protter, M. H. The two noncharacteristic problem with data partly on the parabolic line. Pacific J. Math. **4**, 99–108 (1954).

Le problème considéré est relatif à une équation du type mixte $K(y)u_{xx}+u_{yy}=0$, où K est une fonction croissante deux fois continument différentielle en y telle que $K(0)=0$. Les données sont les valeurs de u le long d'un segment AB de l'axe des x et le long d'un arc de courbe Γ passant par B tel que Γ appartienne au triangle ACB, si AC et BC sont deux caractéristiques concourantes, Γ n'étant coupé qu'en un point au plus par toute caractéristique issue d'un point intérieur au segment AB. Ce problème est classique quand le domaine, frontière comprise, est situé dans la région où l'équation est strictement hyperbolique. La démonstration de l'existence est faite en deux étapes. Tout d'abord on considère le cas où $K(y)$ est une fonction croissante étagée (constante par morceaux); un procédé élémentaire permet alors de calculer la solution. Une analyse plus poussée généralisant des résultats obtenus par l'auteur [M. H. Protter Trans. Amer. Math. Soc. **71**, 416–429 (1951); ces Rev. **14**, 281] permet d'obtenir des bornes relatives aux données de Cauchy le long de $y=0$. La deuxième étape consiste à appliquer les résultats de L. Bers relatifs au problème de Cauchy [NACA Tech. Note. no. **2058** (1950); ces Rev. **12**, 61], ce qui permet de borner à priori la solution dans tout le domaine où elle est définie et, par suite, d'opérer le passage à la limite nécessaire pour traiter le cas où $K(y)$ est une fonction continue (qu'on approche par des fonctions étagées). L'unicité du résultat est prouvée par une méthode directe. *P. Germain* (Paris).

Difference Equations, Special Functional Equations

Samoloff, J. The convergence of the solutions of a class of iterative difference equations. J. Math. Physics **33**, 105–110 (1954).

A real-valued function $f(P)=f(x_1,\cdots,x_k)$ is a weak mean if $\min(x_j)\leq f(P)\leq\max(x_j)$ $(j=1,\cdots,k)$. Let C denote the "cube" $B\leq x_j\leq A$ $(j=1,\cdots,k)$, and suppose f is weak mean on C. Set $\delta=\max(x_j)-\min(x_j)$ and define the separation numbers S, s by

$$S=\inf\left\{\frac{\max(x_j)-f(P)}{\delta}\right\},\quad s=\inf\left\{\frac{f(P)-\min(x_j)}{\delta}\right\}$$

over all $P \in C$ for which $\hat{c} \neq 0$. Then $0 \leqq s, S \leqq 1$ and $s+S \leqq 1$. Then f is called a strong mean on C if $s>0$, $S>0$. The following theorem is proved: Let $f(x_1, \cdots, x_k)$ be a strong mean on C. Then the sequence $\{a_n\}$ defined by

$$a_{n+k}=f(a_n, a_{n+1}, \cdots, a_{n+k-1})$$

converges uniformly for all initial values $(a_0, a_1, \cdots, a_{k-1}) \in C$ to a function $F(a_0, a_1, \cdots, a_{k-1})$ which is uniquely characterized by the properties:

(a) $\lim F(a_0, \cdots, a_{k-1})=x$ as $a_j \to x$ $(j=0, 1, \cdots, k-1)$;

(b) $F(a_0, \cdots, a_{k-1})=F(a_1, \cdots, a_{k-1}, f(a_0, \cdots, a_{k-1}))$.

I. M. Sheffer (State College, Pa.).

Olson, F. R. The non-existence of rational solutions for certain difference equations. Amer. Math. Monthly **61**, 179–181 (1954).

The author presents two simple proofs, one due to himself and the other to L. Carlitz, that the difference equation

$$x^{-k}=F(x+1)-F(x), \quad k=1, 2, \cdots,$$

has no rational solution with coefficients in the complex field. His result remains true if the minus sign on the right hand side of the equation is replaced by a plus sign.

J. M. Danskin (Washington, D. C.).

de Bruijn, N. G. The difference-differential equation $F'(x)=e^{\alpha x+\beta}F(x-1)$. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **56**=Indagationes Math. **15**, 449–458, 459–464 (1953).

The author studies the asymptotic behavior of real solutions of the above equation as $x\to\infty$ for $\alpha>0$, β complex. For real β the problem has previously been investigated by Mahler [J. London Math. Soc. **15**, 115–123 (1940); these Rev. **2**, 133]. The Laplace transform is used to obtain an infinite set of linearly independent solutions, $F_n(x)$, in terms of which any other solution can be expressed. By use of saddle-point techniques, the asymptotic behavior of these functions can be calculated explicitly. The author introduces the adjoint equation and the system biorthogonal to $F_n(x)$, namely $G_n(y)$, and shows that the Green's function has the development $K(y, x)=\sum_n F_n(x)G_n(y)$. Finally, he gives the leading terms in the asymptotic development of arbitrary solutions of the above equation. *R. Bellman.*

Èl'sgol'c, L. È. Approximate methods of integration of differential-difference equations. Uspehi Matem. Nauk (N.S.) **8**, no. 4(56), 81–93 (1953). (Russian)

The author classifies (single) differential-difference equations into advanced, neutral, and retarded types. He surveys various methods for integrating these: the method of successive integration, the method of successive approximations, the method of approximating by Taylor series, and finally the dangerous method of expanding in powers of the log. *J. M. Danskin* (Washington, D. C.).

Cooke, K. L. The asymptotic behavior of the solutions of linear and nonlinear differential-difference equations. Trans. Amer. Math. Soc. **75**, 80–105 (1953).

The author considers the equation

$$(1)\qquad \frac{d}{dt}u(t+1)=a(t)u(t)+b(t)u(t+1)+D[u(t), u(t+1), t],$$

where $D(u, v, t)=\sum_{i+j\geq 2}^{\infty} b_{ij}(t)u^iv^j$. The boundary conditions are given by $u(t)=g(t)$ for $t_0\leqq t\leqq t_0+1$. Equation (1) is supposed to hold for $t>t_0$. The real functions $a(t)$ and $b(t)$ have expansions of the form $a(t)\sim a_0+a_1/t+\cdots$, and $b(t)\sim b_0+b_1/t+\cdots$. The real functions $b_{ij}(t)$ are taken to satisfy $|b_{ij}(t)|\leqq b_{ij}$ for all t. The b_{ij} in turn are such that $\sum b_{ij}u^iv^j$ converges for $|u|$ and $|v|$ sufficiently small. The characteristic equation associated with (1) is

$$(2)\qquad se^s-b_0e^s-a_0=0.$$

This equation has a characteristic root S of largest real part. Let δ be the residue of $(a_1e^{-s}+b_1)/(s-b_0-a_0e^{-s})$ at S.

The main results of the paper are as follows. If equation (1) is linear, i.e. $D(u, v, t)\equiv 0$, there is a unique solution to (1) under the given boundary conditions. If equation (2) has no double roots, then

$$(3)\qquad u(t)=O[t^{\mathrm{Re}(\delta)}e^{t\mathrm{Re}(S)}]$$

as $t\to\infty$. If in the linear equation $a(t)=a_0+a_1/t$ and $b(t)=b_0+b_1/t$, the author is able to give more precise results; the equation under these assumptions is called the equation of the first approximation and it is on this that he begins his investigation. If equation (1) is nonlinear, then there is a constant C such that if $\max_{t\in[t_0, t+1]}|g(t)|\leqq C$, and if $\mathrm{Re}(S)<0$ there is a unique solution satisfying (3). The methods are intricate if standard and are based on the Laplace transform. The results improve those of Bellman [Ann. of Math. (2) **50**, 347–355 (1949); these Rev. **10**, 715] and Wright [Proc. Roy. Soc. Edinburgh. Sect. A. **63**, 18–26 (1950); these Rev. **12**, 106]. *J. M. Danskin.*

Okabe, Jun-ichi. On a forced lateral vibration of a number of particles attached to a string at equal intervals. Rep. Res. Inst. Appl. Mech. Kyushu Univ. **2**, 147–149 (1953).

The nth particle $(n=1, 2, \cdots, N)$ is subject to a harmonic force Q_ne^{ipt}. Assuming the particles also vibrate harmonically with this frequency, one obtains the difference equation

$$\phi_{n-1}+x\phi_n+\phi_{n+1}=Q_n \quad (n=1, 2, \cdots, N),$$

where ϕ_n is the amplitude of the motion of the nth particle, and where x is a parameter. The author solves this by determinants and then evaluates the numerator determinants by means of the method of generating functions. These contain functions of the form of the denominator determinant, which is considered in the review below.

E. Pinney.

Okabe, Jun-ichi. On the roots of the equation

$$\begin{vmatrix} x & 1 & 0 & \cdots & & & \\ 1 & x & 1 & 0 & \cdots & & \\ 0 & 1 & x & 1 & 0 & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & \cdots & 0 & 1 & x \end{vmatrix}=0.$$

Rep. Res. Inst. Appl. Mech. Kyushu Univ. **2**, 150–154 (1953).

The above determinant was the denominator determinant in the solution of the problem in the preceding review. Denoting it by $G(N)$ and expanding by minors,

$$G(N)-xG(N-1)+G(N-2)=0.$$

This difference equation is solved by the method of generating functions in polynomial form. The author notes Rayleigh's Tchebycheff polynomial solution to this equation: $G(N)=\sin(N+1)\theta/\sin\theta$, where $\cos\theta=\frac{1}{2}x$. The author suggests other forms for the solution, but they are all equivalent to Rayleigh's for complex θ. *E. Pinney.*

Integral Equations

Miranda, Carlo. Equazioni integrali con nucleo funzione del parametro: teoria ed applicazioni. Univ. e Politecnico Torino. Rend. Sem. Mat. 12, 67–82 (1953).

A summary of investigations by the author and D. Greco on the integral equation $\varphi(x)=f(x)+\lambda\int G(x, y; \lambda)\phi(y)dy$, where

$$G(x, y; \lambda)=K(x, y)+\lambda H_0(x, y)-\sum_{i=1}^{n}\lambda H_i(x, y)/(\lambda-a_i),$$

$K(x, y)$ symmetric, H_0 and a_iH_i symmetric and positive definite and H_i of finite type. The results summarized are to be found in C. Miranda, Rend. Circ. Mat. Palermo 40, 286–304 (1936), and in D. Greco, Giorn. Mat. Battaglini (4) 2(78), 216–237 (1949); 3(79), 86–120 (1950); 4(80), 102–128 (1951) [these Rev. 11, 437; 12, 103, 709].

T. H. Hildebrandt (Ann Arbor, Mich.).

Vasilache, Sergiu. Le problème de Cauchy et la répartition spectrale des valeurs du paramètre λ, dans la résolution des équations intégro-différentielles. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 4, 7–18 (1952). (Romanian. Russian and French summaries)

The integro-differential equation considered is of the Volterra type, viz.,

$$\sum_{i=0}^{n}H_i(x)\varphi^{(i)}(x)=f(x)+\lambda\int_a^x\sum_{i=0}^{p}K_i(x, y)\varphi^{(i)}(y)dy.$$

A similar equation with $n=0$, and fixed limits has been considered by Bounitzky [Bull. Sci. Math. (2) 32, 14–31 (1908)], equations involving both variable and fixed limits, but not the parameter λ by Andreoli, [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) 22, 2° semestre, 409–414 (1913)]. Following the method of Andreoli, the $\varphi^{(i)}(x)$ for $i<n$, are expressed in terms of $\varphi^{(n)}(x)$ and the equation reduced to the form

$$\text{(A)}\quad H_n(x)\varphi^{(n)}(x)=f(x)+\lambda g(x)+\int_a^x(K_1(x, y)+\lambda K_2(x, y))\varphi^{(n)}(y)dy.$$

From this it follows that if $n\geqq p$, there exists a solution as an entire function of λ; if $n=p-1$, the solution is expressed in a series of rational fractions in λ, valid for λ outside of certain intervals; and for $n\leqq p-2$ the solution is expressible as an entire function in $1/\lambda$. The same theorem is also deduced by integrating the equation n times, expressing $\varphi^{(i)}(x)$ in terms of $\varphi(x)$, and reasoning with an equation in φ similar to (A).

T. H. Hildebrandt.

Radok, Jens Rainer Maria, und Heller, Alfred. Die exakte Lösung der Integralgleichungen gewisser Schwingungsprobleme. Z. Angew. Math. Physik 5, 50–66 (1954).

The paper is concerned with beam vibrations which, with respect to the time t, a space coordinate y and the displacement $z(y, t)$, are set up in the product form $\partial^2z/\partial y^2=e^{i\omega t}Z(y)$. Here $Z(y)$ and ω^2 are the eigenelements of an integral equation $Z(y)=\omega^2\int_0^1K(y, x)Z(x)dx$, where $K(x, y)=K(y, x)$. The kernel K is approximated by a polynomial

$$P_n(x, y)=\sum_{k=0}^{n}a_ky^kx^{n-k}$$

for $x<y$ and by $P_n(y, x)$ for $x>y$. The eigenfunctions $Z(y)$ are approximated by polynomials in y of degree m, and the integral equation after these approximations leads to relations for the coefficients of the approximating polynomial for $Z(y)$. In two cases, where K has already the polynomial form as given by P_n and where the exact solutions are known, it is demonstrated, that the method leads to power expansions of the exact solutions when m goes to infinity. The numerical treatment of integral equations for beam vibrations has already been studied by Lösch [Z. Angew. Math. Mech. 24, 35–41 (1944); these Rev. 7, 489]. He approximates the eigenfunctions by polynomials, which satisfy certain boundary conditions.

H. Bückner.

Germay, R. H. Sur des équations intégro-différentielles récurrentes de forme normale, dont les termes intégraux contiennent les dérivées des fonctions inconnues. Ann. Soc. Sci. Bruxelles. Sér. I. 67, 177–185 (1953).

The set of recurrent integro-differential equations

$$\frac{dy_n}{dx}=F_n[x, y_n(x), y_{n+1}(x); W_{n1}(x), \cdots, W_{np}(x)] \qquad (n=1, 2, \cdots)$$

where the $W_{nj}(x)$ are of the form

$$\int_{x_0}^{x}f_{nj}[x, s; y_n(s), y_{n+1}(s); y_n'(s), y'_{n+1}(s)]ds \qquad (j=1, \cdots, p)$$

is said to be in normal form. If the given functions $F_1, F_2, \cdots, f_{11}, f_{12}, \cdots, f_{np}, \cdots$ possess suitable types of continuity and Lipschitz conditions, then there exists a unique set of solutions satisfying specified initial conditions. A bibliography of nineteen papers, all but two of which are by the author, is given. This comprises the work on this subject for the past twenty years.

I. A. Barnett.

Popovici, Constantin. Sur certaines équations intégrofonctionnelles. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 4, 527–531 (1952). (Romanian. Russian and French summaries)

The paper calls attention to the fact that the functional integral equation $\varphi(z)-\mu\int_a^b\varphi(z+s)ds=ce^{-\lambda z}$ may possess infinitely many solutions since a function of the form $e^{-\lambda z}$ satisfies the homogeneous equation if $\lambda-\mu(e^{-\lambda a}-e^{-\lambda b})=0$. As a consequence a similar thing might be expected for an equation considered by Badescu:

$$\varphi(z)-\int_a^bS(z, s)\varphi(z+s)ds=\sum_{n=1}^{\infty}\psi_ne^{-\lambda_nz},$$

which, in case S is independent of z, has under proper convergence conditions a solution of the form

$$\varphi(z)=\sum_{n=1}^{\infty}\psi_ne^{-\lambda_nz}\Big/\left(1-\mu\int_a^bS(s)e^{-\lambda_ns}ds\right).$$

T. H. Hildebrandt (Ann Arbor, Mich.).

Functional Analysis, Ergodic Theory

Michal, A. D. On bounds of polynomials in hyperspheres and Frechet-Michal derivatives in real and complex normed linear spaces. Math. Mag. 27, 119–126 (1954).

In this paper the author considers the relation between the modulus of real and complex Banach space polynomials and the moduli of their Fréchet differentials. His work is based on well-known theorems due to A. and V. Markoff on the bounds of the derivatives of polynomials of a real

variable, and to S. Bernstein for polynomials of a complex variable. Of considerable interest is the following theorem: The modulus of a homogeneous polynomial in a complex Banach space is equal to the modulus of its polar form. In the case of a real Banach space, this theorem is not true, but bounds relating these moduli are given. The reviewer regrets that he must report his inability to establish the last inequality in the fundamental Lemma 1 and a similar inequality in Lemma 5. *R. G. Bartle.*

Călugăreanu, G. Remarques sur les normes d'un espace vectoriel. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 69–73 (1952). (Romanian. Russian and French summaries)

Proofs of some elementary and well-known facts on such topics as continuity of sub-additive functionals.

V. L. Klee (Seattle, Wash.).

Klee, V. L., Jr. Invariant extension of linear functionals. Pacific J. Math. **4**, 37–46 (1954).

Let $[L, \mathfrak{T}, f, p]$ mean: (i) L is a real linear space; (ii) $\mathfrak{T}$ is a set of linear transformations of L into L; (iii) f is a linear functional on a linear subspace D_f of L; (iv) p is a positively homogeneous subadditive functional on L; (v) $f \leqq p$ on D_f; (vi) $TD_f \subset D_f$ and $fT=f$ for $T \in \mathfrak{T}$. Let $\langle L, \mathfrak{T}, f, p\rangle$ denote the set of linear functionals F on L which agree with f on D_f, such that $F \leqq p$ and $FT=F$, i.e., such that $[L, \mathfrak{T}, F, p]$, $D_F=L$, $F|D_f=f$. The Hahn-Banach theorem asserts $\langle L, \{I\}, f, p\rangle \neq 0$. The author shows $\langle L, \mathfrak{T}, f, p\rangle \neq 0$ is equivalent to (a) $\langle L, \mathfrak{S}, f, p\rangle \neq 0$ for each finite set $\mathfrak{S} \subset \mathfrak{T}$ or (b) $f(x) \leqq p(x+\sum_1^k (T_i - I)y_i)$ for all $x \in D_f$, and all finite sets $\{T_1, \cdots, T_k\} \subset \mathfrak{T}$, $\{y_1, \cdots, y_k\} \subset L$. In case $\mathfrak{T}$ is a semi-group, the above statements are equivalent to (c) $\exists g \in \langle L, \{I\}, f, p\rangle$ such that $gST=gTS$, $gT \leqq p$ for all $S, T \in \mathfrak{T}$.

Let $[L, \mathfrak{T}, f, p]$, $b \geqq 1$ and $pT \leqq bp$ be denoted by $[L, \mathfrak{T}, f, p]_b$. Then if $b=1$ or $p \geqq 0$, the following are sufficient conditions for $\langle L, \mathfrak{T}, f, bp\rangle \neq 0$: (d) $\mathfrak{T}$ is an abelian semi-group; (e) every finite subset of $\mathfrak{T}$ is contained in a finite subgroup of $\mathfrak{T}$; (f) p is a norm for L and $\mathfrak{T}$ is a group, compact in the uniform topology of operators. These and other corollaries are given together with applications to finitely additive extensions of planar Lebesgue measure and fixed point theorems in Banach spaces. *B. Gelbaum.*

Gurevič, L. A. On unconditional bases. Uspehi Matem. Nauk (N.S.) **8**, no. 5(57), 153–156 (1953). (Russian)

The author chooses the following as his definition for unconditional convergence in a Banach space E: $\sum x_n$ converges unconditionally if and only if $\sum |f(x_n)| < \infty$ for each $f \in E$. (The more customary definition is: $\sum x_n$ converges unconditionally if and only if $\sum x_{\varphi(n)}$ converges for every 1-1 map φ of the integers onto themselves. The definitions are equivalent in case E is weakly sequentially complete.) A basis is called unconditional in case the expansion of each element of E converges unconditionally. The principal theorem of the author is: $\{x_n\}$ is an unconditional basis for E if and only if there is a constant K for which $|\sum_1^n \pm \xi_i x_i| \leqq K|\sum_1^n \xi_i x_i|$, for any distribution of $\pm$ and any sequence of real numbers $\{\xi_i\}$. Obvious equivalent formulations are given. In the space c_0 (of null sequences $\{a_n\}$ with $|\{a_n\}| = \sup |a_n|$) the author gives an example similar to one given by the reviewer [Duke Math. J. **17**, 187–196 (1950); these Rev. **11**, 729] of a basis which is not unconditional. A necessary condition for an unconditional basis in Hilbert space is also given. *B. Gelbaum.*

Loomis, L. H. Linear functionals and content. Amer. J. Math. **76**, 168–182 (1954).

Given a linear space L of real functions and a non-negative linear (n.n.l.) functional I on L, there is a "two-sided completion" L^I, the class of functions (on the same set) on which all n.n.l. functionals which agree with I on L, coincide. Let E be the linear envelope of the characteristic functions which belong to L^I. The first problem of integral representation (relative to the finitely additive set function $\mu(e)=I(\chi_e)$ where $\chi_e \in L^I$ is the characteristic function of the set e) is to determine when E^I (the integrable functions) includes a given function f of L. When L is a vector lattice, it suffices that $1 \in L$ and f be bounded. Neither of these alone suffices. To treat the general case a notion of improper integration is introduced, for which L^I undergoes a further process of "one-sided completion." The details will not be noted here. The results readily yield the more familiar integral representation theorems. [The date on reference [4] should be 1952.] *R. Arens* (Princeton, N. J.).

Hewitt, Edwin. Remark on orthonormal sets in $\mathfrak{L}_2(a, b)$. Amer. Math. Monthly **61**, 249–250 (1954).

Burgess, D. C. J. Abstract moment problems with applications to the l^p and L^p spaces. Proc. London Math. Soc. (3) **4**, 107–128 (1954).

The author gives an extension of results centering around the classical Hausdorff moment problem to abstract spaces. The elements μ_n of the sequence are supposed to belong to a complex reflexive Banach space $\mathfrak{X}$ (the separability assumption in Theorem 3.13 was later found to be unnecessary). The discussion requires introducing a weak Lebesgue-Stieltjes integral $'\int_E g(t)dx_t$ where $x_t \in \mathfrak{X}$ is of weakly bounded variation and $f[' \int_E g(t)dx_t] = \int_E g(t)df(x_t)$ for every $f \in \mathfrak{X}^*$. The author also gives a representation of the integral $\int_E x_t dt$ when $x_t \in L_p[0, R]$ for $t \in E$; he proves a Helly theorem (a sequence $\{x_t^n\}$ of functions of uniformly weakly bounded variation on $[a, b]$ with $\{x_a^n\}$ bounded contains a subsequence converging weakly to a function x_t of weakly bounded variation), a couple of weak compactness theorems and representation theorems for linear operations on $L_p[0, 1]$ to $\mathfrak{X}$ or from $\mathfrak{X}^*$ to L_p. These tools are then applied to abstract moment problems. Necessary and sufficient conditions are found for the existence of solutions of the three problems

$$\mu_n = {}'\int_0^1 t^n dx_t; \quad \mu_n = \int_0^1 t^n x_t dt; \quad f(\mu_n) = \int_0^1 t^n \frac{d}{dt}[f(x_t)]dt.$$

In particular, the author applies his criteria to the problem of representing a double sequence of complex numbers as moments of a sequence of functions in l_p or a sequence of complex-valued functions as moments of the Laplace or the Stieltjes transform of a kernel in L_p. *E. Hille.*

Berman, D. L. On translational linear trigonometric polynomial operations. Doklady Akad. Nauk SSSR (N.S.) **92**, 693–694 (1953). (Russian)

Let E_1 and E_2 be spaces of type E [Berman, same Doklady (N.S.) **88**, 9–12 (1953); these Rev. **14**, 767] of functions defined on $(-\infty, \infty)$. The author states that certain linear operators U which commute with translation must be of the form

$$U(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)T(x-t)dt,$$

T being a trigonometric polynomial of order n. A proof is sketched. *E. Hewitt* (Seattle, Wash.).

Plans Sanz de Bremond, Antonio. Essai d'une algèbre linéaire infinie en le champ des matrices bornées. Collectanea Math. **5**, 285–329 (1952). (Spanish. French summary)

Linear forms $\sum a_i x_i$ are considered which converge for all $\{x_i\} \in \mathfrak{H}$ (real Hilbert space). Linear dependence is defined for an infinite sequence of forms, both in the case that the coefficient matrix is bounded, and in general; and systems of infinitely many equations in infinitely many unknowns are treated. In particular, these general considerations are related to various subclasses of the class of bounded matrices that were discussed by the author earlier [Revista Acad. Ci. Madrid **46**, 273–302 (1952); these Rev. **14**, 768]. Part of the work parallels certain sections of Julia, "Introduction mathématique aux théories quantiques" [partie 2, Gauthier-Villars, Paris, 1938]. *I. M. Sheffer.*

S.-Nad', Bèla [Sz.-Nagy, Béla]. On conjugate cones in Hilbert space. Uspehi Matem. Nauk (N.S.) **8**, no. 5(57), 167–168 (1953). (Russian)

The author reproduces a short proof [Acta Sci. Math. Szeged **12**, Pars A, 228–238 (1950); these Rev. **12**, 266] of a theorem published by Tagamlickiĭ [Uspehi Mat. Nauk (N.S.) **7**, no. 2(48), 180–183 (1952); these Rev. **14**, 184]. *E. Hewitt* (Seattle, Wash.).

Kostyučenko, A., and Skorohod, A. On a theorem of N. K. Bari. Uspehi Matem. Nauk (N.S.) **8**, no. 5(57), 165–166 (1953). (Russian)

N. Bari showed that if $\{\varphi_n\}$ and $\{\psi_n\}$ are orthonormal systems in Hilbert space, and if $\sum|\varphi_n-\psi_n|^2<\infty$, then both systems are complete if one is. The authors reprove this theorem in a neat and straightforward manner. The result is applied to the proof that the orthonormal solutions of certain Sturm-Liouville systems are complete. *B. Gelbaum* (Minneapolis, Minn.).

Taldykin, A. T. On the existence of characteristic values and on the completeness of systems of characteristic elements for linear operators. Doklady Akad. Nauk SSSR (N.S.) **92**, 1121–1124 (1953). (Russian)

We say that a bounded linear operator, defined on a Hilbert space, has property CV if it has a characteristic value in its Fredholm circle. If A is such an operator, then the following conditions are sufficient, and the first is necessary, for it to have this property: 1) for some f, $\limsup \|A^n f\|^{1/n} \geqq 1/r_A$, where r_A is the Fredholm radius of A; 2) there exists a C with the property CV, and $r_A{}^n\|A^n f\| \geqq r_C{}^n \|C^n f\|$; 3) there exists B, which commutes with A, such that $C=AB$ has property CV, while $r_C\|B\| \leqq r_A$; 4) there exists B, permutable with A, such that $\|B\| \leqq M\|A\|$, $C=A+B$ has property CV, and $r_C(1+M) \leqq r_A$; 5) $A=A_1+A_2$, either $A_1A_2=0$ or $A_2A_1=0$, and the operator on the right of the zero product has property CV.

If $A=\sum_{k=-m}^{\infty}(\lambda-\lambda_0)^k c_k$, all elements of the form $c_{-k}h$ $(k=1, 2, \cdots, m)$ will be called elements corresponding to A: if $k=m$; they are characteristic elements for A.

The following conditions are sufficient (and the first is necessary) for the elements corresponding to A^* (for all λ) to form a complete set: 6) for any f in H,

$$\limsup \|A^n f\| \geqq 1/r_A;$$

7) there is a C whose corresponding elements form a complete set and, for any f, $r_A{}^n\|A^n f\| \geqq r_C{}^n\|C^n f\|$.

The hypotheses of 3), 4) and 5) give sufficient conditions for the set corresponding to A to be complete, provided that the conditions that the operators C or A_2 have the property CV are replaced by the conditions that the set corresponding to them is complete. *J. L. B. Cooper* (Cardiff).

Devinatz, A. A note on semi-groups of unbounded self-adjoint operators. Proc. Amer. Math. Soc. **5**, 101–102 (1954).

In this note the author extends the representation theorem for semi-groups of bounded self-adjoint operators due originally to E. Hille [Proc. Nat. Acad. Sci. U. S. A. **24**, 159–161 (1938)] and Sz. Nagy [ibid. **24**, 559–560 (1938)]. The following theorem is proved. Let $[T_x; x>0]$ be a semi-group of self-adjoint operators acting on a Hilbert space, i.e. $T_{x_1}T_{x_2}=T_{x_1+x_2}$. Further suppose for all $f \in \bigcap_{x>0} D(T_x)$, that $(T_x f, f)$ is either a bounded or measurable function of x in some interval. Then there exists a unique resolution of the identity $[E_t]$ such that $E_t=0$ for $t<0$ and $T_x=\int_0^\infty t^x dE_t$. Here $D(A)$ denotes the domain of the operator A. *R. S. Phillips* (New Haven, Conn.).

Orihara, Masae. Correction to my paper: "Rings of operators and their traces." Mem. Fac. Sci. Kyūsyū Univ. A. **8**, 89–91 (1953).

See same Mem. A. **5**, 107–138 (1950); these Rev. **13**, 756.

Tomita, Minoru. On rings of operators in non-separable Hilbert spaces. Mem. Fac. Sci. Kyūsyū Univ. A. **7**, 129–168 (1953).

L'auteur n'est pas au courant des travaux récents sur la question (sa bibliographie est antérieure à 1950). Les chaps. 1, 2, 3 ne sont pas nouveaux (le théorème attribué en 2.1 au rapporteur est inexact: l'application ♮ canonique d'un anneau d'opérateurs fini n'est pas faiblement continue). Le chapitre 4 contient une présentation plus originale de la réduction des anneaux d'opérateurs: soient $\mathfrak{H}$ un espace hilbertien, Z un anneau d'opérateurs abélien, Ω son spectre, μ une "mesure complètement additive" sur Ω. L'auteur définit: 1) pour tout $\lambda \in \Omega$, un espace hilbertien $\mathfrak{H}(\lambda)$ (complété du quotient d'un sous espace de $\mathfrak{H}$ variable avec λ); 2) pour tout $f \in \mathfrak{H}$, un champ de vecteurs $\lambda \rightarrow f(\lambda)$ défini sauf sur un ensemble rare; 3) pour tout $A \in Z'$, un champ d'opérateurs $\lambda \rightarrow A(\lambda)$ essentiellement borné; tout ceci avec les propriétés habituelles; 4) pour tout anneau d'opérateurs M de centre Z, un champ d'anneaux d'opérateurs $\lambda \rightarrow M(\lambda)$; en général, $M(\lambda)$ et $M'(\lambda)$ ne sont pas des facteurs tels que $M'(\lambda)=M(\lambda)'$; mais, sans hypothèse de séparabilité, l'auteur parvient à établir ce résultat en compliquant un peu la méthode de réduction (la définition des $\mathfrak{H}(\lambda)$ fait intervenir un sous espace fixe partout dense bien choisi de $\mathfrak{H}$); l'auteur étudie les relations entre la classe des $M(\lambda)$ et celle de M, et la décomposition des fonctions-poids. Toutefois, ces résultats supposent M semi-fini; d'autre part, il n'y a pas de méthode bien nette pour reconstruire $\mathfrak{H}, A, M$ à partir des $\mathfrak{H}(\lambda)$, $A(\lambda)$, $M(\lambda)$ supposés donnés. *J. Dixmier.*

Pukánszky, L. On a theorem of Mautner. Acta Sci. Math. Szeged **15**, 145–148 (1954).

The author discusses and modifies a result of the reviewer [Ann. of Math. (2) **51**, 1–25; **52**, 528–555 (1950); these Rev. **11**, 324; **12**, 157] on direct integrals of Hilbert spaces and the corresponding decompositions of rings of operators and unitary representations. *F. I. Mautner.*

Pukánszky, L. The theorem of Radon-Nikodym in operator-rings. Acta Sci. Math. Szeged **15**, 149–156 (1954).

Two results on the representation of linear functionals on operator rings are proved. The first of these is stated to be

Despite the author's apparent assumption of strong continuity, the intension was only to assume strong continuity relative to the unit sphere, and his proofs make & use only of the latter (weaker) assumption.

equivalent to Th. **14** of Segal, Ann. of Math. (2) **57**, 401–457 (1953) [these Rev. **14**, 991] and the second is due originally to Dye [Trans. Amer. Math. Soc. **72**, 243–280 (1952); these Rev. **13**, 662]. Actually, the first theorem is not equivalent to the cited one, in which the indefinite integral of a positive integrable operator is characterized, in that the author's assumption of strong continuity for this functional (in place of strong continuity relative to the unit sphere) materially limits the class of functionals treated (as is clear, e.g., in the case of the ring of all bounded operators). The proof of Dye's theorem establishing the existence of the derivative of a strongly continuous positive linear functional with respect to another such functional under the assumption of absolute continuity is based in part on arguments of Dye, as noted by the author. *I. E. Segal.*

Edwards, R. E. On functions which are Fourier transforms. Proc. Amer. Math. Soc. **5**, 71–78 (1954).

Let G, $\hat{G}$ be dual locally compact Abelian groups, $\overline{\mathfrak{M}}$ the set of bounded Radon measures on $\hat{G}$. A C-algebra is defined to be a commutative complete normed complex algebra for which there is a constant $K>0$ such that $\|f^2\|>K\|f\|^2$ for all elements f of the algebra. The principal theorem is that if there exists an algebraic isomorphism of a C-algebra R onto a subalgebra of $\overline{\mathfrak{M}}$, then R is of finite dimension.

As consequences extensions of theorems of Segal and Hewitt are proved. Among these results are the following. If every continuous function on G which tends to zero at infinity and which vanishes outside a subset A of G is a $T(\overline{\mathfrak{M}})$, i.e., a Fourier transform of an element of $\overline{\mathfrak{M}}$, then the interior of A is finite. If every continuous uniformly almost periodic function on G which has its spectrum in a subgroup $\hat{H}$ of $\hat{G}$ is a $T(\overline{\mathfrak{M}})$, then $\hat{H}$ is finite. If there is a neighbourhood U of the unit in G such that every continuous function on G, zero outside U, is in $T(\overline{\mathfrak{M}})$, then G is discrete.

J. L. B. Cooper (Cardiff).

Matsushita, Shin-ichi. Positive linear functionals on self-adjoint B-algebras. Proc. Japan Acad. **29**, 427–430 (1953).

The author states a number of results for Banach algebras with an involution. Some of the results are incorrect.

J. A. Schatz (Bethlehem, Pa.).

Wolfson, Kenneth G. The algebra of bounded operators on Hilbert space. Duke Math. J. **20**, 533–538 (1953).

Let K be a B^*-algebra; i.e., K is a complex Banach algebra with an involutory anti-automorphism * which satisfies the conditions $(\lambda x)^*=\bar{\lambda}x^*$ and $\|x^*x\|=\|x\|^2$ for $x \in K$ and λ a complex number. If T is a subset of K, then the set of all $x \in K$ such that $xT=(0)$ is called a "left annulet." The author obtains the following characterization of the algebra $B(H)$ of all bounded operators on a Hilbert space H: Let K be a B^*-algebra which contains an identity and minimal right ideals; then K is isomorphic (in a norm and *-preserving manner) to an algebra $B(H)$ for some Hilbert space H if, and only if, (1) K contains a smallest closed (non-zero) 2-sided ideal and (2) if J_1, J_2 are left annulets which satisfy $J_1J_2^*=(0)$, then J_1+J_2 is a left annulet. It is shown that H is essentially unique for K. The following result is also obtained: Let $E(H)$ be a dual ring [Kaplansky, Ann. of Math. (2) **49**, 689–701 (1948); these Rev. **10**, 7] of bounded operators on a Hilbert space H which is closed in the uniform topology and is primitive [Jacobson, Amer. J. Math. **67**, 300–320 (1945); these Rev. **7**, 2]; then $E(H)$ is the ring of all completely continuous operators on H. The point here is that $E(H)$ is not assumed to be self-adjoint or even closed under multiplication by scalars.

C. E. Rickart (New Haven, Conn.).

Wolfson, Kenneth G. The algebra of bounded functions. Proc. Amer. Math. Soc. **5**, 10–14 (1954).

Let K be a commutative B^*-algebra with an identity. It is shown that K is isomorphic (in a norm and adjoint preserving manner) to an algebra $B(X)$ of all bounded complex-valued functions on an essentially unique set X if and only if (1) every non-zero closed ideal of K contains a minimal ideal and (2) the sum of two annulets is an annulet. By an annulet in K is meant the annihilator of some subset of K. The set X is essentially unique in the sense that if K is isomorphic to $B(X_1)$ and $B(X_2)$ then there is a lattice isomorphism of the lattice of all subsets of X_1 upon the lattice of all subsets of X_2. An analogous result is shown to hold where K is a p-ring and, instead of $B(X)$, the set of all functions on a set X to $GF(p)$ is considered. *B. Yood.*

Burgess, D. C. J. Tauberian theorems in a Banach lattice, with applications to the L^p spaces. Proc. Cambridge Philos. Soc. **50**, 242–249 (1954).

This is a continuation of the author's paper in Proc. London Math. Soc. (3) **3**, 378–384 (1953) [these Rev. **15**, 136] where he considered Laplace transforms with values in an arbitrary Banach space X. In the present paper X is supposed to be a Banach lattice, x_t is a function of strongly bounded variation having values in X and $y_s=\int_0^\infty e^{-st}dx_t$ is considered for real s only. Among the results we note: If y_s converges for $s>0$ and $y_s \to x'$ as $s \downarrow 0$, then the condition $\bar{y} \geqq \theta$, $t\bar{y}+\int_0^t u dx_u$ is nondecreasing in $0<t<\infty$ implies that $x_{t+0} \to x'$ as $t \to \infty$. In particular, if $x_t=\sum_{0<n<t}a_n$ and $na_n \geqq -\bar{y}$, $\bar{y} \geqq \theta$, then $\lim_{s\to+0}y_s=x'$ implies $\sum_1^\infty a_n=x'$. The results are applied to mean convergence of Laplace-Stieltjes transforms in $L_p[0, \infty]$, $1 \leqq p$, with the partial ordering relation $\phi(\cdot) \geqq \psi(\cdot)$ if and only if $\phi(t) \geqq \psi(t)$ for almost all t. Set $f_T(s)=\int_0^T e^{-st}dg(t)$ and $f_\infty(s)=f(s)$, where the integral is supposed to converge for $s>0$. Suppose that $f_1(s)$ and $f(s) \in L_p[0, \infty]$. Then a sufficient condition that

$$\operatorname*{l.i.m.}_{T\to\infty} f_T(s)=f(s)$$

is the existence of a positive $\phi(\cdot) \in L_p[0, \infty]$ such that

$$(T_2-T_1)\phi(s)+\int_{T_1+}^{T_2} te^{-st}dg(t) \geqq 0$$

almost everywhere for $T_2>T_1 \geqq 1$. In particular, if

$$f(s)=\sum_1^\infty a_n e^{-ns} \in L_p[0, \infty]$$

and there exists a positive $\phi(\cdot) \in L_p[0, \infty]$ such that $na_n e^{-ns}+\phi(s) \geqq 0$ almost everywhere, then $f(s)$ is the L_p-limit of its partial sums. *E. Hille* (New Haven, Conn.).

Calculus of Variations

Conti, Roberto. Sulla semicontinuità degli integrali del calcolo delle variazioni in forma ordinaria. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15**, 149–157, 158–164 (1953).

In the first of these notes the author recalls the theorem of L. Tonelli [Fondamenti di calcolo delle variazioni, vol. 1,

Zanichelli, Bologna, 1922] that lower semicontinuity of the integral $I(C)=\int_C f(x, y, y')dx$ with respect to uniform convergence in the class of curves C absolutely continuous in the sense of Tonelli (ACT) implies that $f(x, y, z)$ is convex in z. The author then proves that if the class of admissible curves is enlarged to include all continuous curves of bounded variation in the sense of Tonelli (BVT) then lower semicontinuity of $I(C)$ implies that $f(x, y, z)\equiv P(x, y)$. He proves an analogous result for functionals involving higher derivatives, and states without proof the corresponding result for functionals $\int\int_D f(x, y, z, p, q)dxdy$. He illustrates this last statement by noting that the Dirichlet integral is not lsc (lower semi-continuous) in the class of BVT continuous functions though it is in the class of ACT functions. The basis of his proofs is the Cantor function, which he uses to approximate uniformly an arbitrary continuous function on a closed interval by a BVT function having an arbitrary constant derivative almost everywhere.

In the second note he considers a kind of convergence defined as follows: $y(x)\rightarrow y_0(x)$ if (i) $y(x)$ converges uniformly to $y_0(x)$ and (ii) $y'(x)$ converges in the mean of order 1 to $y_0'(x)$. Here $C\rightarrow C_0$ means that if C_0 is represented by $y_0(x)$ and the C by $y(x)$ then $y(x)\rightarrow y_0(x)$. If $I(C)$ is lsc with respect to "$\rightarrow$" he calls it a weakly lsc functional. Modelling his proof on a theorem in Tonelli's book referred to above, he proves that if $f(x, y, z)>\lambda-\Lambda|z|$ for (x, y) sufficiently close to C_0, where $\Lambda\geqq 0$, then $\lim\inf_{C\rightarrow C_0} I(C)\geqq I(C_0)$. From this it follows, for example, that if $f(x, y, z)$ is convex in z then $I(C)$ is weakly lsc in the class of continuous BVT functions. *J. M. Danskin* (Washington, D. C.).

Aruffo, Giulio. Sul differenziale generalizzato delle forme continue e su un'estensione del lemma di Haar. Ricerche Mat. **2** (1953), 241–265 (1954).

The author obtains several ways of characterizing those differential forms of degree $r+1$ which are differentials (in what he terms the generalized sense) of forms of degree r, and he extends to them a known construction of a local primitive. By these means he establishes corresponding extensions of the familiar lemmas of Schauder and Haar in the calculus of variations. Another proof of Haar's lemma is given by Miranda [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **85**, 246–254 (1951); these Rev. **14**, 46].

L. C. Young (Madison, Wis.).

De Sloovere, H. Le calcul des variations successives d'une intégrale multiple, par la méthode invariantive de Th. De Donder. I, II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 474–480, 948–952 (1953).

Basing his notation on that of Th. De Donder [Théorie invariantive du calcul des variations, nouvelle éd., Gauthier-Villars, Paris, 1935], the author develops formulae for the first, second and third variations of a multiple integral, obtaining a law of recurrence for the integrands in the successive variations. The varied integral is $\int_{(n)}\mathfrak{F}d(x^1\cdots x^n)$ with $\mathfrak{F}=\mathfrak{F}(x^i, y^\alpha, y_i^\alpha)$ in part I and $\mathfrak{F}=\mathfrak{F}(x^i, y^\alpha, y_i^\alpha, y^\alpha_{ij})$ in part II, the subscripts denoting partial derivatives.

J. L. Synge (Dublin).

Cooperman, Philip. On a variational problem having a third order differential equation as a necessary condition for an extremum. Proc. Amer. Math. Soc. **5**, 309–310 (1954).

Theory of Probability

Richter, Hans. Zur Grundlegung der Wahrscheinlichkeitstheorie. II. Axiomatik der Erwartungskoeffizienten. Math. Ann. **125** (1952), 223–234 (1953).

Richter, Hans. Zur Grundlegung der Wahrscheinlichkeitstheorie. III. Die Begründung des Additions- und des Multiplikationssatzes. Math. Ann. **125**, 335–343 (1953).

Richter, Hans. Zur Grundlegung der Wahrscheinlichkeitstheorie. IV. Wahrscheinlichkeitsrechnung. Math. Ann. **126**, 362–374 (1953).

[For part I see Math. Ann. **125**, 129–139 (1952); these Rev. **14**, 484.] Parts II–IV present and explore an axiomatic treatment of probability based on the two intuitive notions of degree of certainty and physical closure. Some idea of the final axiom system, presented in part III, can be obtained from the following list of formally undefined terms: experimental scheme, H; experiment under H, $\bar{H}$; product of two experimental schemes, $F(H_1, H_2)$; outcomes possible under H, x_k; expectation coefficient of a set E of x_k's under H, $P(E|H)$.

It is a theorem that some monotonic function of P satisfies not only the axioms but also the additive and multiplicative laws of probability, suitably interpreted. A fifth part on statistical inference is forthcoming. *L. J. Savage.*

★Schulz, Günther. Wahrscheinlichkeitsrechnung und mathematische Statistik. Naturforschung und Medizin in Deutschland, 1939–1946, Band 3. Angewandte Mathematik, Teil I, pp. 185–198. Verlag Chemie, Weinheim, 1953. DM 20.00.

Alda, Václav. A note on Poisson's distribution. Čehoslovack. Mat. Ž. **2**(77), 243–246 (1952). (Russian. English summary)

For each n let $x_n=\sum_j x_{nj}$ be a sum of mutually independent random variables with means 0, and variances which approach 0 uniformly when $n\rightarrow\infty$ and have sum 1. Then the author proves that x_n has the Poisson distribution with mean value 1 as limiting distribution when $n\rightarrow\infty$ if and only if $\sum_j(x^2_{nj}-x_{nj})\rightarrow 1$ in probability. *J. L. Doob.*

Lisçu, Traian. Sur les moments stochastiques des moments de sélection. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 791–799 (1952). (Romanian. Russian and French summaries)

Let $x_1, \cdots, x_n$ be mutually independent random variables with a common distribution function, and let $u_n=\sum_1^n u(x_j)/n$, $v_n=\sum_1^n v(x_j)/n$. The author finds expressions for $E\{u_n^r v_n^s\}$ in terms of more elementary moments. *J. L. Doob.*

Andersen, Erik Sparre. On sums of symmetrically dependent random variables. Skand. Aktuarietidskr. **36**, 123–138 (1953).

Results of the author [Skand. Aktuarietidskr. **32**, 27–36 (1949); these Rev. **11**, 256] and D. A. Darling [Proc. Amer. Math. Soc. **2**, 511–517 (1951); these Rev. **13**, 258] are generalized. The chance variables $X_1, X_2, \cdots$ are called symmetrically dependent if, for any n, the distribution function of $X_1, \cdots, X_n$ is symmetric in all its arguments. An event is a Borel set in the space of $x_1, \cdots, x_n$ defined by restrictions on $x_1, \cdots, x_n$; if an event is invariant under permutations of $x_1, \cdots, x_n$ it is called by the author symmetric with respect to $X_1, \cdots, X_n$. Let $S_n=\sum_{i=1}^n X_i$

$S_0=0$. Let

L_n: the index $k(=0, 1, \cdots, n)$ for which
$S_k=\max(S_0, S_1, \cdots, S_n)$ and
$S_k>\max(S_0, S_1, \cdots, S_{k-1})$,

M_n: the index $k(=0, 1, \cdots, n)$ for which
$S_k=\min(S_0, S_1, \cdots, S_n)$ and $S_k<\min(S_{k+1}, \cdots, S_n)$,

N_n: the number $k(=0, 1, \cdots, n)$ of sums
$S_0, S_1, \cdots, S_n$, which are strictly positive.

The author proves a number of theorems of which we cite two: Theorem 1. Let $X_1, X_2, \cdots$ be symmetrically dependent random variables. Let C_n be an event, which is symmetric with respect to $X_1, \cdots, X_n$. Then

$$\Pr\{[N_n=m]C_n\}=\Pr\{[L_n=m]C_n\}=\Pr\{[M_n=n-m]C_n\},\quad m=0, 1, \cdots, n.$$

Theorem 2. If $X_1, X_2, \cdots$ is a sequence of identically distributed, independent random variables and K_n is one of the variables L_n, M_n or N_n, then

$$\Pr\{K_n=m\}=\Pr\{K_m=m\}\Pr\{K_{n-m}=0\},\quad \text{for } n=1, 2, \cdots \text{ and } m=0, 1, \cdots, n.$$

J. Wolfowitz (Ithaca, N. Y.).

Fisz, M. The limiting distributions of sums of arbitrary independent and equally distributed r-point ($r\geq 2$) random variables. Bull. Acad. Polon. Sci. Cl. III, 1, 235–238 (1953).

For each n, let X_n be a sum of n independent random variables with a common distribution, which may depend on n. It is supposed that the distributions of the summands are concentrated at $r\geqq 2$ points. Here the points, but not r, may depend on n. The following result is announced without proof. If centering and scaling constants can be found which make X_n have a limit distribution when $n\rightarrow\infty$, this distribution is necessarily a sum of $\leqq r-2$ independent variables of Poisson type and (possibly) a normal variable, or else is a sum of $r-1$ independent random variables of Poisson type. (Here a random variable x is said to be of Poisson type if for some constants a, b, the random variable $ax+b$ has a Poisson distribution.) Various special cases are considered in which it is possible to specialize this result further. *J. L. Doob* (Urbana, Ill.).

Diananda, P. H. The central limit theorem for m-dependent variables asymptotically stationary to second order. Proc. Cambridge Philos. Soc. 50, 287–292 (1954).

In a previous paper [same Proc. 49, 239–246 (1953); these Rev. 14, 771] the author proved the asymptotic normality of the sum of stationary, m-dependent random variables with finite variance. In the present paper this result is shown to hold when the assumption of stationarity is replaced by the condition that for every p, the covariance of X_n and X_{n+p} converges to a constant as $n\rightarrow\infty$, provided that a condition of the Lindeberg type is satisfied. Extensions to sums of random vectors are given

W. Hoeffding (Chapel Hill, N. C.).

Rvačeva, E. L. On the maximum discrepancy between two empirical distributions. Ukrain. Mat. Žurnal 4, 373–392 (1952). (Russian)

Gnedenko and Korolyuk [Doklady Akad. Nauk SSSR (N.S.) 80, 525–528 (1951); these Rev. 13, 570] have shown how the problem of comparing the empirical distributions of two equal samples can be reduced to a random walk problem. Further results were obtained by Gnedenko and the present another [ibid. 82, 513–516 (1952); these Rev. 13, 760]. The reader is referred to either of these reviews for notations and explanations. The author obtains new and detailed results. Let $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ be the two samples, and let $(z_1, z_2, \cdots, z_{2n})$ be their rearrangement in increasing order. Let $\omega_n(j)=F_1(z_j)-F_2(z_j)$, where F_1 and F_2 are the two empirical distributions. Denote by $D_n^+(p, q)$, $D_n^-(p, q)$, respectively, the maximum and minimum of $\omega_n(j)$, when $0\leqq p\leqq j\leqq q\leqq 2n$. The author obtains the joint conditional distribution of the k-dimensional random variable $\{D_n^+(0, 2n), D_n^+(1, 2n), \cdots, D_n^+(k, 2n)\}$ for given values of $\omega_n(0), \omega_n(1), \cdots, \omega_n(k)$, and also the corresponding unconditional distribution. Next she derives analogous conditional and unconditional distributions for the $2k$-dimensional variable

$$\{D_n^+(0, 2n), D_n^-(0, 2n), \cdots, D_n^+(k, 2n), D_n^-(k, 2n)\}.$$

In each case the limiting distribution as $n\rightarrow\infty$ is obtained. Finally the distribution of the maximal term of the sequence $\{\omega_n(j)\}$ is given together with its limiting form. As an interesting corollary one gets that the probability that the sequence $\{\omega_n(j)\}$ assumes its maximum at one and only one place j with $0<j<2n$ equals one half; and that each place j has the same probability $1/2(2n-1)$ to be this place of maximum. *W. Feller* (Princeton, N. J.).

Širokorad, B. V. On the applicability of the central limit theorem to Markov chains. Izvestiya Akad. Nauk SSSR. Ser. Mat. 18, 95–104 (1954). (Russian)

Let s_n be the sum of the first n variables of a Markov chain with the two states 0, 1 and mth transition matrix $[{}_mp_{ij}]$, where ${}_mp_{10}\sim am^{-\alpha}$, ${}_mp_{01}\sim bm^{-\beta}$, $0<\alpha\leqq\beta\leqq 1$. The following theorems are proved. (1) If $\beta<1$, then $[s_n-E\{s_n\}]/(\text{st. dev. } s_n)$ is asymptotically normal, with mean 0 and variance 1. (2) If $\alpha<\beta=1$, then s_n/n^a has asymptotically the distribution with density const.$\times t^{(b/a)-1}e^{-at}$, for $t\geqq 0$. (3) If $\alpha=\beta=1$, then s_n/n has asymptotically the distribution with density const.$\times t^{b-1}(1-t)^{a-1}$, for $0\leqq t\leqq 1$.

J. L. Doob (Urbana, Ill.).

Yuškevič, A. A. On limit theorems connected with the concept of entropy of Markov chains. Uspehi Matem. Nauk (N.S.) 8, no. 5(57), 177–180 (1953). (Russian)

Let $x_1, x_2, \cdots$ be the random variables of a stationary Markov chain with finitely many states, and transition matrix $[p_{ij}]$. It is supposed that there is positive probability of going from any state to any other state. The author then proves two results used in information theory. He does not observe that the results can be based on direct applications of the law of large numbers and central limit theorem to sums of the form $\sum_1^n p_{x_ix_{i+1}}$. *J. L. Doob* (Urbana, Ill.).

Mihoc, G. La loi des événements rares pour les chaînes de Markoff. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 4, 783–790 (1952). (Romanian. Russian and French summaries)

The author finds the characteristic function of the asymptotic limiting distribution of the number of occurrences of a specified state in n trials of an m state Markov chain with stationary transition probabilities. Here $n\rightarrow\infty$ and the transition probabilities vary in a certain way with n. Koopman [Trans. Amer. Math. Soc. 70, 277–290 (1951); these Rev. 14, 1100] treated the case $m=2$. *J. L. Doob.*

Itô, Kiyosi. Stochastic differential equations in a differentiable manifold. II. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **28**, 81–85 (1953).

In a previous paper [Nagoya Math. J. **1**, 35–47 (1950); these Rev. **12**, 425] the author solved stochastic differential equations on a differentiable manifold to get Markov processes on the manifold with an assigned generating operator, that is, assigned diffusion coefficients. In the present paper, the same problem is solved for a more general generating operator, by reduction to the methods used in the previous case. See K. Yosida [Osaka Math. J. **5**, 65–74 (1953); these Rev. **15**, 36] and S. Itô [ibid. **5**, 75–92 (1953); these Rev. **15**, 36] for related results obtained using differential equation methodology. *J. L. Doob.*

Hopf, Eberhard. The general temporally discrete Markoff process. J. Rational Mech. Anal. **3**, 13–45 (1954).

Let $(X, \mathfrak{B}, \lambda)$ be a probability space, viz. a triple of an abstract set X, a countably additive family $\mathfrak{B}$ of subsets $A \subseteq X$ such that $X \in \mathfrak{B}$ and a probability λ in X countably additive on $\mathfrak{B}$. Let $L^1(X)$ be the Banach space of $\mathfrak{B}$-measurable functions integrable with respect to λ. The author has succeeded in proving the following individual ergodic theorem, extending the results due to G. D. Birkhoff [Proc. Nat. Acad. Sci. U. S. A. **17**, 656–660 (1931)], S. Kakutani [Proc. Imp. Acad. Tokyo **16**, 49–54 (1940); these Rev. **1**, 343] and J. L. Doob [Trans. Amer. Math. Soc. **63**, 393–421 (1948); these Rev. **9**, 598]. Let S be a linear operator on $L^1(X)$ to $L^1(X)$ satisfying the conditions: (1) $(Sf)(x)$ is non-negative and $\int_X (Sf)(x)\lambda(dx) = \int_X f(x)\lambda(dx)$; (2) $S\cdot 1 = 1$. Then $\lim_{n\to\infty} n^{-1}\sum_{k=0}^{n-1}(S^k f)(x)$ exists λ-almost everywhere.

The proof is carried out operator-theoretically [cf. the reviewer, Jap. J. Math. **17**, 31–36 (1940); these Rev. **2**, 105; N. Dunford and D. S. Miller, Trans. Amer. Math. Soc. **60**, 538–549 (1946); these Rev. **8**, 280]. It makes use of the mean ergodic theorem in Banach spaces, a convergence theorem of Banach [Bull. Sci. Math. (2) **50**, 36–43 (1926)], and a maximal ergodic theorem which the author proves for the linear operator S^* satisfying (1): (3) For any $f \in L^1(X)$ and for any integer m, we have

$$\lambda(A) \geq 0 \quad \text{for} \quad A = \left\{x;\ \sup_{n \leq m} \sum_{k=0}^{n-1} (S^{*k}f)(x) \geq 0\right\}.$$

The ergodic structure of the process corresponding to this operator S^* is also studied; its dissipative and conservative part and the S^*-invariant sets. A conjecture is stated about the influence of the dissipative part upon the conservative part. *K. Yosida* (Osaka).

***Landé, Alfred. Probability in classical and quantum theory.** Scientific papers presented to Max Born, pp. 59–64. Hafner Publishing Co. Inc., New York, N. Y., 1953. $2.50.

Vaulot, Emile. Délais d'attente des appels téléphoniques dans l'ordre inverse de leur arrivée. C. R. Acad. Sci. Paris **238**, 1188–1189 (1954).

A new hypothesis for order of service, namely "last come, first served", in telephone delay systems is studied. The system otherwise is of the classical Erlang specification: x trunks, pure chance (Poisson) arrivals, exponential holding (service) times, no defections in the waiting line; the results are for statistical equilibrium conditions. The function formulated, $G_n(t)$, is the probability of further delay at least t of a call which is $n+1$ in line; thus $G_0(t)$ is the probability of delay at least t of any (delayed) call. With y calls arriving in unit time (which as usual is the average holding time), the following differential recurrence relation is given:

$$dG_n(t)/dt = xG_{n-1}(t) - (x+y)G_n(t) + yG_{n+1}(t).$$

This is solved subject to the boundary conditions $G_n(0) = 1$, $n \geq 0$, and $G_{-n}(t) = 0$, all n; the solution is given as a definite integral of trigonometric and exponential functions and is said to be also expressible by means of the Bessel I functions. Reviewer's note: The last also appear in solutions of similar differential recurrences considered by Bateman [Bull. Amer. Math. Soc. **49**, 494–512 (1943); these Rev. **5**, 71] but Bateman's boundary conditions are different. Finally, some remarks are made on the relation of these and allied results to service in arbitrary order. *J. Riordan.*

Meidell, Birger. Randbemerkungen zum Landréschen Maximum. Skand. Aktuarietidskr **36**, 168–181 (1953).

The assumption of a Pareto law for the distribution of sums at risk is used in the determination of a life insurance company's maximum sum for retention (i.e. without reinsurance). *H. L. Seal* (New York, N. Y.).

Mathematical Statistics

Steinhaus, Hugo. Table of shuffled four-digit numbers. Rozprawy Mat. **6**, 46 pp. (1954). (Polish, Russian and English)

This is a table of the 10000 four-digit numbers 0000–9999 arranged in a supposedly random way. Since the table contains each number once and only once it can be used for sampling problems in which samples, once drawn, are not replaced. Thus it could be used for a schedule for the retirement of bonds. By reading the digits vertically by fours, instead of horizontally, one obtains an ordinary random number table in which numbers may be missing or may occur more than once. No tests have been made of the actual randomness obtained. The table was produced by hand by a series of randomizing transformations applied to a table of 100 rows and columns. *D. H. Lehmer.*

Grab, Edwin L., and Savage, I. Richard. Tables of the expected value of $1/X$ for positive Bernoulli and Poisson variables. J. Amer. Statist. Assoc. **49**, 169–177 (1954).

The authors give five place tables for

$$(1)\quad E(1/X|n, p) = (1-q^n)^{-1}\sum_{x=1}^{n}\binom{n}{x}x^{-1}p^x q^{n-x} \quad (q = 1-p)$$

for $n = 2(1)20$; $p = .01, .05(.05).95, .99$ and $n = 21(1)30$; $p = .01.05(.05).50$ as well as for

$$(2)\quad E(1/X|m) = e^{-m}(1-e^{-m})^{-1}\sum_{x=1}^{\infty} m^x/(x!x)$$

for

$m = .01, .05(.05)1.0(.1)2.0(.2)5.0(.5)7.0(.1)10(2)20.$

These tables can be used to determine mean and variance of a sum of independently and identically distributed random variables in case the sample size (number of terms in the sum) is also a random variable and is independently distributed from the observations with either a positive Bernoulli or a positive Poisson distribution. *E. Lukacs.*

de Misès, Richard. Théorie et application des fonctions statistiques. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 11 (1952), 374–410 (1953).

"Dans ce qui suit je donne un bref rapport sur une série des conférences que j'ai tenues, sur l'invitation de M. le professeur Francesco Severi, à l'Istituto di Alta Matematica de l'Université de Rome en février 1952. Dans mes leçons j'utilisais les résultats de plusieurs de mes publications antérieures, en suppléant des lacunes, en modifiant et, parfois, rectifiant les démonstrations. Mes principaux travaux sur les fonctions statistiques sont les suivants: Rev. Fac. Sci. Univ. Istanbul (N.S.) 1, no. 1, 61–80 (1935); Monatsh. Math. Phys. 43, 105–128 (1936); Ann. Inst. H. Poincaré 6, 185–212 (1936); Ann. Math. Statistics 18, 309–348 (1947)" [ces Rev. 9, 194]. (From the author's introduction.) *J. Wolfowitz* (Ithaca, N. Y.).

Ulin, Bengt. An extremal problem in mathematical statistics. Skand. Aktuarietidskr. 36, 158–167 (1953).

Let C be the class of density functions which are non-decreasing in $(-\infty, \alpha)$, non-increasing in (α, ∞), and have mean zero and variance σ^2. The author considers the problem treated by H. L. Selberg [Skand. Aktuarietidskr. 23, 114–120 (1940); these Rev. 2, 228] of finding

$$J=\inf \int_{-\xi}^{\xi} f(x)dx=\inf I(f)$$

with respect to all f in C, and of finding $g(x) \in C$ (if it exists) such that $I(g)=J$. He shows that, except for some special cases, a minimizing density function $g(x)$ exists and is a step-function with at most 4 jumps. For a number of values of ξ, α, σ a table of $g(x)$ and J is provided. The best minorant of J, independent of α, is given. *J. Wolfowitz.*

Cohen, A. C., Jr. Estimation of the Poisson parameter from truncated samples and from censored samples. J. Amer. Statist. Assoc. 49, 158–168 (1954).

The author gives maximum likelihood estimates for the parameter of a Poisson population if only truncated or censored samples are available. The censoring is of type I, i.e. observations above or below fixed truncation points are censored. Censoring of type II (omission of the $n-k$ smallest or largest observations) is not considered.
E. Lukacs (Washington, D. C.).

Seelbinder, B. M. On Stein's two-stage sampling scheme. Ann. Math. Statistics 24, 640–649 (1953).

Consider a sequence of variables independently distributed according to a normal distribution with mean μ and standard deviation σ. A two stage test of the hypothesis $|\mu-\mu_0|<d$ which has power independent of the variance σ^2 was devised by C. Stein [same Ann. 16, 243–258 (1945); these Rev. 7, 213]. The total number n of observations to be taken in the two samples has a distribution which depends on the level of significance α, the ratio d/σ, and the size n_0 of the first sample. The present paper gives tables of the expectation of n for $\alpha=.1, .05, .02, .01$, for a range of values of d/σ from .01 to 1.0 and a range of values of n_0 from 5 to 240. An approximation based on the normal distribution is shown to be adequate if $n_0 \geqq 60$. *L. LeCam.*

Chand, Uttam. On the derivation and accuracy of certain formulas for sample sizes and operating characteristics of nonsequential sampling procedures. J. Research Nat. Bur. Standards 47, 491–501 (1951).

For testing a hypothesis H_1 about the distribution of the chance variable X against a given alternative H_2, formulas giving the number of independent observations on X required to achieve given probabilities of type I and type II errors are given for several common problems. The formulas are for the most part approximations based on the fact that the test statistics are approximately normally distributed. The cases discussed include testing the hypotheses that parameters of binomial, Poisson, or normal distributions have specified values; and testing the hypotheses that corresponding parameters of two binomial, Poisson, or normal distributions are equal. In formula (1), the symbol μ_2 should be replaced by μ_1. *L. Weiss* (Charlottesville, Va.).

Birnbaum, Allan. Admissible tests for the mean of a rectangular distribution. Ann. Math. Statistics 25, 157–161 (1954).

Explicit characterizations are given of the minimal complete class and a minimal essentially complete class of tests of a simple hypothesis specifying the mean of a uniform distribution of known range. Examples are given of tests which are optimal against various alternatives.
L. J. Savage (Chicago, Ill.).

Choudhury, P. A note on testing of normality. Science and Culture 19, 453–454 (1954).

Berlyand, H. L., and Kvit, I. D. On a problem of comparison of two samples. Dopovidi Akad. Nauk Ukrain. RSR 1952, 13–15 (1952). (Ukrainian. Russian summary)

The Kolmogorov-Smirnov theorems describe the asymptotic distributions of various measures of the discrepancies between two empirical distribution functions for large samples. Kvit [Doklady Akad. Nauk SSSR (N.S.) 71, 229–231 (1950); these Rev. 11, 528] derived the corresponding asymptotic distributions when considering the empirical distribution functions only on certain central subintervals of their intervals of variation. In the present article the authors state without proof the corresponding results when considering the empirical distribution functions only outside these subintervals. *J. L. Doob* (Urbana, Ill.).

Jiřina, Miroslav. Sequential estimation of distribution-free tolerance limits. Čehoslovack. Mat. Ž. 2(77), 221–232 (1952); correction 3 (78), 283 (1953). (Russian. English summary)

The following sequential procedure for finding tolerance limits $A<B$ for a continuous distribution function $F(z)$ is investigated. Let $r>0$, $s>0$, $k>0$. During the first stage, take $r+s$ observations and set $A^{(1)}=z_{(r)}$, $B^{(1)}=z_{(r+1)}$, where $z_{(u)}$ is the uth order statistic. During the jth stage, $j=2, 3, \cdots$, continue taking observations as long as

$$(*) \qquad A^{(j-1)} \leqq z_{t+i} \leqq B^{(j-1)}$$

and $i<k$, where t is the number of observations taken during the preceding $j-1$ stages. If (*) holds also for $i=k$, terminate the procedure and set $A=A^{(j-1)}$, $B=B^{(j-1)}$. If $z_{t+i}<A^{(j-1)}$ or $z_{t+i}>B^{(j-1)}$, $i\leqq k$, set $A^{(j)}=z_{(r)}$ and $B^{(j)}=z_{(t+i+1-s)}$. It is shown that the procedure terminates with probability 1 and that

$$\alpha=P\left\{\int_A^B dF(z)\geqq\beta\right\}=(1-\beta)^{r+s}\exp\left[(r+s)\sum_{m=1}^{k}\beta^m/m\right].$$

This expression can be used to find minimum k corresponding to given values of r, s, α, β. This has been done for $r=s=1$ and certain values of α and β close to 1. By considering also $r=0$ (or $s=0$) and modifying the procedure correspondingly,

one-sided tolerance limits are obtained. It is shown that if $r+s>1$ the sequential procedure is better than the corresponding fixed-sample procedure for values of β close to 1, in the sense that the probability α for the sequential procedure is greater than the corresponding probability for the fixed sample procedure. For $r+s=1$, the fixed sample procedure is always better than the sequential procedure.

G. E. Noether (Boston, Mass.).

Johnson, N. L. Some notes on the application of sequential methods in the analysis of variance. Ann. Math. Statistics 24, 614–623 (1953).

C. M. Stein and M. A. Girshick [unpublished] have shown that the Wald sequential probability ratio test can be applied to problems involving composite hypotheses. The author applies these ideas to certain problems included under the general linear hypothesis. Special attention is given to the standard one-way classification problem, Model I and Model II. For Model II (components of variance) the author has computed tables of acceptance and rejection numbers for $\alpha=\beta=.05$, for $\alpha=\beta=.01$ and for various group sizes, thus reducing the test to a straight-forward calculation. Alternative procedures are discussed which either make use of these tables or are such that Wald's average sample number formula can be applied.

M. Sobel.

Bechhofer, Robert E. A single-sample multiple decision procedure for ranking means of normal populations with known variances. Ann. Math. Statistics 25, 16–39 (1954).

The practical limitations of classical analysis of variance tests (model I) have been pointed out on numerous occasions. The experimenter, after performing a test (and rejecting the hypothesis) of equality of means of normal populations, frequently selects that population as "best" whose sample mean is "best" without knowing the probability involved.

An alternative meeting this objection is presented. If the experimenter will (or can) specify in advance the smallest differences in means worth detecting for a particular type of ranking, e.g., the population with the largest (smallest) mean, and the probability with which he desires to make a correct ranking (when these are the true differences), then from tables given here, published or to be published, he may determine the appropriate sample sizes provided the population variances are known. The probabilities of correct rankings are expressible in terms of multivariate normal integrals. A simplified lower bound for such a probability is found by using a "least favorable configuration" of population means. Tables based on such a lower bound are thus conservative.

H. Teicher (Lafayette, Ind.).

Wagner, Gustav. Folgetest für die Abnahmeprüfung von Mengen mit grossen und kleinen Stückzahlen. Mitteilungsblatt Math. Statist. 5, 89–102 (3 plates) (1953).

The first part of the paper is an exposition of the underlying principles of quality control, followed by a presentation of the sequential test for attributes when the population sampled is indefinitely large. The latter part of the paper considers the same test when the population sampled is not so large. Let N = size of population, p = proportion defective, m = number in sample, s_m = number of defective items in sample. When $p_0<p_1\leqq 0.1$, the boundaries of the region of continued inspection in the (m, s_m)-plane can be approximated by two lines that intersect when $m=N$. The lot is rejected whenever $s_m=Np_0$. As $N\to\infty$ this polygonal region of continued inspection tends to the well known strip between two parallel lines.

S. W. Nash.

Kanô, Seigo. On the filter problem of a stationary stochastic process. Bull. Math. Statist. 5, no. 3–4, 47–51 (1953).

The author provides a kernel which minimizes the variance of the error in predicting values of a function in terms of two given functions, its cross correlation with one of them, and the autocorrelation of all three.

A. Blake.

da Jager, J. Sampling distributions and graduations. Verzekerings-Arch. Actuarieel Bijvoegsel 31, 29*–50* (1954).

q_x, the relative frequency of death between ages x and $x+1$, is distributed binomially. Thus the means and variances of numerous actuarial functions of q_x can be calculated, at least approximately. Examples are: deferred temporary annuity values and the statistically estimated constants of a Makeham graduation (viz., $-\ln(1-q_x)=\alpha+\beta c^x$).

H. L. Seal (New York, N. Y.).

✶**Stumpers, F. L. A bibliography of information theory. Communication theory—cybernetics.** Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Mass., 1953. i+i+46 pp.

Mathematical Economics

✶**Motzkin, T. S., Raiffa, H., Thompson, G. L., and Thrall, R. M. The double description method.** Contributions to the theory of games, vol. 2, pp. 51–73. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. **$4.00.**

This paper presents a computational scheme for the solution of a finite zero-sum two-person game, and for the solution of finite systems of linear inequalities. It is based, as its title suggests, on the double description of a convex polyhedral cone as the intersection of a finite number of closed half-spaces or as the set of non-negative linear combinations of spanning vectors. The spanning vectors may be considered as the basis of a parametric solution of the set of homogeneous inequalities defining the half-spaces. If another half-space (inequality) is added to the system, the new basis consists of all vectors from the old basis that lie in this half-space combined with certain combinations of all pairs of vectors in the old basis that lie on opposite sides of the hyperplane that bounds the half-space.

"The idea was first used in the dissertation of T. S. Motzkin [Dissertation, Basel, 1933, Jerusalem, 1936] for purely theoretical purposes. Independently, Raiffa, Thompson, and Thrall developed the corresponding computational scheme for the case of finite games. The exposition of this scheme is given in Parts I and II. Again independently, the computational scheme for general systems of linear inequalities was given by T. S. Motzkin. This procedure forms the content of Part III." (From the authors' introduction.)

H. W. Kuhn (Bryn Mawr, Pa.).

Malmquist, Sten. Index numbers and indifference surfaces. Trabajos Estadistica 4, 209–242 (1953). (Spanish summary)

A summary, elucidation, and extension of the theory of cost-of-living index numbers due initially to Konyus,

Frisch, Roy, Staehle, Lerner and others. One section considers so-called Divisia-indexes and shows that only for very special time paths of prices and quantities will a Divisia index coincide with a "true" cost-of-living index. The author leans heavily on the concept of price-indifference surfaces introduced by Hotelling and further developed by Roy; these are loci of (relative) prices for which a maximizing consumer is just able to attain a given level of satisfaction. This leads to a simple characterization of a price index: let P_0 and P_1 be the price vectors to be compared. Let α be a scalar such that αP_1 is indifferent to P_0. Then the price index for period 1 on period 0 as base, and with respect to the real income level of period 0 is $1/\alpha$. Similar considerations lead to the Staehle and other inequalities which provide limits to the price index. An analogous index is defined in the commodity space and it is shown that a partial factor-reversal property is satisfied by the price and quantity indexes. The author then considers the situation where some commodities are effectively rationed. A sample result is the following: If the rations are the same for periods 0 and 1 and the consumer buys his entire rations, then the Staehle inequalities still hold, even though the consumed quantities do not represent maximum budgets in the ordinary sense. Other inequalities are proved. The conclusion of Theorem 1 should read: $I_{01}(\mu_0)=0D/0C$. *R. Solow.*

Allais, M. Puissance et dangers de l'utilisation de l'outil mathématique en économique. Econometrica 22, 58–71 (1954).

Herstein, I. N. Some mathematical methods and techniques in economics. Quart. Appl. Math. 11, 249–262 (1953).

This paper performs the useful service of presenting some aspects of pure mathematics being applied currently to problems in economics. Among the methods and problems discussed in some detail are a derivation of the Slutsky equation via the calculus, a problem in Welfare Economics treated by the theory of convex sets, matrix theory as applied to international trade, and a game-theoretical approach to the personnel assignment problem. Many other subjects are touched lightly and cited in an interesting bibliography. *H. W. Kuhn* (Bryn Mawr, Pa.).

***Stone, Richard. The measurement of consumers' expenditure and behaviour in the United Kingdom, 1920–1938. Vol. I.** Assisted by D. A. Rowe, W. J. Corlett, Renée Hurstfield, and Muriel Potter. Cambridge, at the University Press, 1954. xl+448 pp. $18.50.

This is Volume I in a series designed to carry back to the years before 1938 the time series needed for the construction and analysis of the social accounts of the U.K. The tremendous research effort represented was begun in 1941. After a sweetly reasonable methodological introduction, 16 chapters describe the basic estimates: annual quantities consumed, average prices, and expenditure on various categories of foods, alcoholic drinks, tobacco, rents rates and water, fuel and light, for 1920–38. These are especially noteworthy for the full description of the estimating procedures, for the careful (subjective) evaluation of the reliability of the series, and for the ingenuity of the methods, which often come as close to creating something out of nothing as the laws of thermodynamics will allow.

Chapters 20–23 present and discuss estimated demand functions for these various commodities. Income elasticities and the effect of family size are estimated from budget studies, leaving own- and other-price elasticities and residual trends to be estimated from the time series. The method is single-equation linear regression using first differences of logarithms. As elsewhere in the volume, the fullness of the presentation is exceptional and the useful information contained is great. But this is not the place to review in detail.

Chapter 17 is a brief and more or less nontechnical review of some of the purposes and problems of econometrics. There is an elementary discussion of tests of significance which rather slurs over the crucial notions of alternative hypotheses and power. A lucid if loose introduction to the problem of identification in linear systems is given by means of the usual simple examples. The subsequent use of straightforward least squares is pragmatically justified by (1) the added labor of simultaneous estimation; (2) the relatively large sampling variability to be expected when the estimate of one parameter is made to depend on many others; (3) the difficulties introduced by errors of observation in the predetermined variables; (4) the difficulty of finding really predetermined variables at all; and (5) the possibility that there may be estimation problems more serious than those which can be avoided by structural estimation.

Chapter 18 is about the pure theory of consumer choice, which provides the theoretical framework for the subsequent empirical work. Stone begins by sketching the theory of choice for a consumer with an (ordinal) utility defined only over a discrete (and finite, but this is apparently inessential) collection of points in the n-dimensional commodity space. The point is to take account of the indivisibility of many commodities. Equivalence classes are defined in the obvious way: Naturally these are not indifference surfaces; Stone suggests some substitute concepts, but since they are derived from the partial ordering of "domination" and not from the preference ordering, they are rather unsatisfactory. It is indicated that most of the qualitative properties of demand functions which can be deduced in the usual smooth case also hold here. Little seems to be gained by this reformulation. The next section runs quickly over the standard theory of the continuous case with some emphasis on the verifiable properties of demand functions, and is followed by a survey of the formal relations between individual and market demand parameters, as worked out by Staehle, Marschak, Haavelmo, the author and others. These usually take the form that some market parameter is a weighted average of corresponding individual parameters with quantities bought as weights. In the case of market income elasticities, the distribution of income must be brought in explicitly. After some inconclusive remarks about irreversibilities, changes in taste, and other time-aspects of demand, and a statement of some theorems of Tobin and Houthakker on demand elasticities under rationing, the chapter concludes with a statement of the formulation to be used in the subsequent empirical analyses. The Engel curves and the demand functions are constant-elasticity type; in the latter, consumption per equivalent adult is made to depend on real income and relative prices. Where possible the income elasticity is taken from the budget study and the time series corrected for estimated income effect. An exponential time trend is also included. In general the estimates are subject to the theoretically appropriate constraint of zero-degree homogeneity in money income and prices. A test of this hypothesis on the data confirms it.

Chapter 19 on "Estimation Problems and Statistical Procedures" begins with a clear exposition of the classical

least squares regression model, in delightfully compact matrix notation. Stone then takes up four ways in which the assumptions which lend a certain optimality to least squares may (and are likely to) fail in economic time series. These are (1) serial correlation of the residuals, (2) the existence of simultaneous relations which make the residuals and right-hand side variables statistically dependent, (3) errors of measurement, and (4) collinearity. Since the methods and results described are available in the literature no detail need be given here. Under (1) Stone uses the test of Durbin and Watson on regression residuals and for estimation purposes prefers the idea of autoregressive transformation of the data, which leads to the use of first differences as suggested by Cochrane and Orcutt. Under (2) "the simultaneous equations complication has been ignored for reasons which are, on the whole, of a practical character". Under (3) there is an interesting exposition of the method of instrumental variates of Reiersøl and Geary in terms of simultaneous estimation. This idea is attributed to Durbin. Here and under (4) there is a very brief and mostly formal description of Frisch's bunch maps (used profusely in the empirical work). The chapter concludes with a useful study of the effect of using "extraneous estimates" (i.e., using an estimate based on one set of observations in a regression analysis based on other observations, as the income elasticities from budget studies are used in the time series analyses) also due to Durbin, and some remarks on estimating trends. The whole chapter is an extremely valuable economist's-eye view of least squares, but formal rather than statistical in approach.

To sum up, this book is a mine and model of careful empirical and statistical work. It also contains some interesting theoretical material, and shows a nice interplay between the two. And it must surely be one of the most sumptuously printed and published scholarly works of modern times.

R. Solow (Cambridge, Mass.).

TOPOLOGY

Dirac, G. A. The structure of k-chromatic graphs. Fund. Math. **40**, 42–55 (1953).

The author continues his investigation of k-chromatic graphs [J. London Math. Soc. **27**, 85–92 (1952); Math. Z. **54**, 347–353 (1951); these Rev. **13**, 572, 672]. In this paper he derives some properties of those sets of edges or vertices whose removal decomposes a k-chromatic graph into two or more connected parts. *W. T. Tutte* (Toronto, Ont.).

Errera, Alfred. Sur les polyèdres de genre zéro. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11** (1952), 315–322 (1953).

The author suggests a method of classifying those spherical polyhedra in which just three faces meet at each vertex. The method is based on a theorem of H. Whitney [Ann. of Math. (2) **32**, 378–390 (1931)] according to which on any such polyhedron in which no three faces constitute a multiply connected region a simple closed curve can be drawn so as to avoid the vertices and pass just once through each of the faces. *W. T. Tutte* (Toronto, Ont.).

Padmavally, K. On a characterization of minimally bicompact spaces: corrections and additions. J. Indian Math. Soc. (N.S.) **17**, 143–149 (1953).

It is shown that in a bicompact T_1-space, one point at least, but not always each point, has convergence character; and that the topology may be enlarged (open sets added) retaining bicompactness, and giving every point convergence character, perhaps isolating some points. The statement "if each closed set A, for each limit point p of A has a subset A not containing p of which p is the sole complete limit point then the space will lose bicompactness if open sets are adjoined to its topology" [made in K. Padmavally, same J. (N.S.) **16**, 63–68 (1952); these Rev. **14**, 571] is disproved by examples. *R. Arens* (Princeton, N. J.).

van Est, W. T., und Freudenthal, Hans. Vollständige Regularität und Normalität in ihrer Bedeutung für ein Kompaktheitskriterium. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 409–411 (1953).

In earlier papers [same Proc. **54** = Indagationes Math. **13**, 295–296, 369–370 (1951); these Rev. **13**, 372], the authors considered the following properties of a topological space R. $E_1(E_2)$: If F is an arbitrary set of real continuous functions on R such that $\{f_1, f_2\} \subset F$ implies $\max(f_1, f_2) \in F$ and $f \in F$ implies $\inf f = 0$, then there is a sequence x_i in R with $\lim f(x_i) = 0$ for each $f \in F$ (there is a point $x \in R$ with $f(x) = 0$ for each $f \in F$). They proved that for completely regular R, E_2 is equivalent to compactness of R, and claimed to prove that E_1 is equivalent to E_2. That the latter need not be true is demonstrated in the present paper by examples of non-compact completely regular R having the property E_1. (Such examples have also been constructed by J. Novák and S. Mrówka, and will appear in Fund. Math.) It is proved, however, that for normal R, E_1 is equivalent to E_2, and hence to compactness of R. *V. L. Klee.*

Katětov, M. Correction to "On real-valued functions in topological spaces" (Fund. Math. 38 (1951), pp. 85–91). Fund. Math. **40**, 203–205 (1953).

Correction of an error in the paper cited [these Rev. **14**, 304]. *E. Hewitt* (Seattle, Wash.).

Novák, J. On the Cartesian product of two compact spaces. Fund. Math. **40**, 106–112 (1953).

Two regular countably compact spaces are constructed whose Cartesian product is not countably compact. If X and Y are, however, regular countably compact spaces and if X is compact, then $X \times Y$ is countably compact (Katětov). *E. Hewitt* (Seattle, Wash.).

Novák, Josef, and Mišík, Ladislav. On L-spaces of continuous functions. Mat.-Fyz. Sbornik Slovensk. Akad. Vied Umení **1**, 1–17 (1951). (Slovak. Russian and French summaries)

Let L be an $\mathfrak{L}$-space in the sense of Fréchet. Let $\{x_{m,n}\}_{m,n=1}^{\infty}$ be a double sequence of points in L. A diagonal subsequence of this double sequence is any sequence of the form $\{x_{m_i,n_i}\}_{i=1}^{\infty}$ in which no index m_i appears an infinite number of times. A point $x \in L$ is said to have property ρ if there exists a double sequence $\{x_{m,n}\}_{m,n=1}^{\infty}$ such that $\lim_{m\to\infty} x_{m,n} = x$ for all n but such that no diagonal subsequence of the double sequence $\{x_{m,n}\}_{m,n=1}^{\infty}$ converges to x. A construction is given showing that the space of continuous functions on [0, 1] under pointwise convergence contains a point having property ρ. Two $\mathfrak{L}$-spaces L_1 and L_2 of which L_1 contains a point with property ρ have the property that $L_1 \times L_2$ is not an $\mathfrak{L}$-space. Let G be an Abelian group which is an $\mathfrak{L}$-space in

which the group operation is compatible with the limit operation. The property $A^{--}=A^{-}$ obtains for all subsets A of G if and only if G contains no point with property ρ.
E. Hewitt (Seattle, Wash.).

Mišík, Ladislav. Concerning a property of the space of polynomials defined on the interval $\langle 0, 1)$. Čehoslovack. Mat. Ž. 2(77), 233–237 (1952). (Russian. English summary)

Let G be as in the preceding review. Suppose that there is a metric δ defined in G, in addition to the limit operation already postulated, with the following property. For every sequence $\{x_n\}_{n=1}^{\infty}$ of points in G for which there exists a sequence $\{y_n\}_{n=1}^{\infty}$ converging in the $\mathfrak{L}$-sense to a certain point x and for which $\lim_{n\to\infty}\delta(x_n, y_n)=0$, the sequence $\{x_n\}_{n=1}^{\infty}$ also converges to x in the $\mathfrak{L}$-sense. Suppose that there is a subset A of G for which $A^{--}\neq A^{-}$. If P is a subgroup of G dense in the δ-sense, then P contains a subset B such that $B^{--}\neq B^{-}$. The example $G=C([0, 1])$ and $P=$all polynomials with rational coefficients is a case in point.
E. Hewitt (Seattle, Wash.).

Vilhelm, Václav, and Vitner, Čestmír. Continuity in metric spaces. Časopis Pěst. Mat. 77, 147–173 (1952). (Czech)

This is a partly expository article. A novelty is the introduction of Cauchy continuity: a mapping f of a metric space X with metric ρ onto a metric space Y with metric σ is said to be Cauchy continuous if $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in Y whenever $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. It is noted that uniform continuity implies Cauchy continuity which implies continuity. A large number of theorems are proved, of which the following may be taken as typical. The mapping f is uniformly continuous if and only if the conditions $A, B\subset X$ and $\rho(A, B)=0$ imply $\sigma[f(A), f(B)]=0$. Elegant proofs are given, for metric spaces, of Urysohn's lemma and Urysohn's extension theorem. It is shown that Cauchy continuity and uniform continuity can be preserved in applying Urysohn's extension theorem. *E. Hewitt.*

McShane, E. J. A theory of convergence. Canadian J. Math. 6, 161–168 (1954).

Exposé de la théorie de la convergence en topologie générale, qui, à part la terminologie, ne contient absolument rien de nouveau. La seule différence avec les définitions usuelles (telles qu'elles sont par exemple données dans Bourbaki, Topologie générale [Hermann, Paris, 1940; ces Rev. 3, 55]) est que pour définir la limite d'une fonction f suivant une base de filtre $\mathfrak{B}$, l'auteur ne suppose pas que f prenne ses valeurs dans un espace topologique, mais seulement dans un ensemble filtré; mais, comme il le remarque lui-même, cela ne change rien d'essentiel à la théorie.
J. Dieudonné (Evanston, Ill.).

Obreanu, Filip. Sur un théorème de Baire. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 4, 285–290 (1952). (Romanian. Russian and French summaries)

There is obtained the following extension of a well-known theorem of Baire: Suppose E is a topological space, m is a cardinal number, and $(A, >)$ is a directed set of cardinality $\leqq m$. Suppose E' is a completely regular space having a compatible uniformity base of cardinality $\leqq m$. Suppose $\{f_\alpha: \alpha \in A\}$ is a family of continuous functions on E to E', pointwise convergent (in terms of the ordering $>$) to the function f. Then the set of points at which f is discontinuous can be covered by m nowhere dense subsets of E.
V. L. Klee (Seattle, Wash.).

Barbalat, I. Limites multiples dans un espace uniforme. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 4, 311–317 (1952). (Romanian. Russian and French summaries)

Some straightforward extensions of a well-known result of Pringsheim on convergence of double sequences.
V. L. Klee (Seattle, Wash.).

Nakamura, Masahiro. On a lemma of Sunouchi and Yano. Kōdai Math. Sem. Rep. 1953, 127–128 (1953).

The author observes that in a compact Hausdorff space S, if $T(x)$ is a homeomorphism with $T^n(a)$ dense in S for each $a \in S$, there is for each open set O a constant k such that for any $x \in S$, $T^n(x) \in O$ with $n\leqq k$. *P. Civin.*

Bertolini, Fernando. A proposito di una mia osservazione sulla nozione di connessione. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 11, 123–124 (1952).

One of the examples given in a former paper of the author [same Rend. (5) 9, 74–78 (1950); these Rev. 12, 518] did not suit its purpose, and hence is replaced by another example here. *A. Rosenthal* (Lafayette, Ind.).

Knaster, B., et Urbanik, K. Sur les espaces complets séparables de dimension 0. Fund. Math. 40, 194–202 (1953).

An orderly presentation is given of results, mostly new, concerning the topological structure and classification of 0-dimensional G_δ-sets in the Hilbert cube (i.e., the 0-dimensional separable complete spaces). If C denotes the Cantor perfect set on an interval I and B_u denotes the set of all left (right) hand end points of segments complementary to C on I, proofs are given that: (1) every G_δ-set of dimension 0 is homeomorphic to a closed subset of C decreased by the omission of a subset of B_u; (2) every dense in itself 0-dimensional G_δ-set is homeomorphic with C minus some subset of B_u; (3) if D is any countable subset of C, $C-D$ is homeomorphic with C minus some subset of B_u; (4) every scattered set is homeomorphic with a subset of a compact well ordered subset of B_u; (5) every 0-dimensional G_δ-set is homeomorphic with the set of all end points of a dendron of order $\leqq 3$. The authors point out that (1) is already known, (2) is a reformulation of a result of Mazurkiewicz, and that (3) is, of course, a corollary to (2). Also (5) effects solutions to problems previously proposed by Čech and Kuratowski.
G. T. Whyburn (Charlottesville, Va.).

Knaster, B., et Reichbach, M. Sur la caractérisation topologique de l'ensemble des bouts d'une courbe. Fund. Math. 40, 13–28 (1953).

By a well known result of Menger's, the set of all end points (i.e., points of order 1) of a continuum is a 0-dimensional G_δ-set. The authors consider the question as to whether the property of being a 0-dimensional G_δ-set is equivalent to the property of being homeomorphic with the set of all end points of some curve (i.e. some 1-dimensional continuum). It is shown that given any 0-dimensional G_δ-set B, there exists a curve C in the plane the set of all end points of which is homeomorphic with B. The question as to whether C can also be chosen so as to be (a) locally connected or (b) a dendrite is considered in another paper (see the preceding review). *G. T. Whyburn.*

Knaster, B., et Reichbach, M. Un lemme sur les F_σ. Fund. Math. 40, 172–179 (1953).

The authors extend a theorem of Menger's concerning the decomposition of closed sets in a compact metric space

X into arbitrarily small closed sets with restricted intersections. It is shown that every F_σ set of dimension $n \geqq 0$ in such a space X can be represented as the (infinite) union of a sequence (F_i) of arbitrarily small compact sets with diameters going to 0 with $1/i$ and such that the intersection of any r of these sets, $r=1, 2, \cdots, n+2$, is of dimension $\leqq n-r+1$. Corollaries and applications of this result are discussed. *G. T. Whyburn* (Charlottesville, Va.).

Lubański, M. An example of an absolute neighbourhood retract, which is the common boundary of three regions in the 3-dimensional Euclidean space. Fund. Math. **40**, 29–38 (1953).

The example constructed here not only has the property described in the title but also has the property that for every $\epsilon>0$, the set can be expressed as the union of a finite number of absolute retracts, each of diameter $<\epsilon$. In addition, it is pointed out that the construction can be modified to yield a common boundary of $m>3$ domains. The construction is quite complicated. *E. G. Begle.*

Kinoshita, Shin'ichi. On some contractible continua without fixed point property. Fund. Math. **40**, 96–98 (1953).

A widely held conjecture, that every compact contractible set must have the fixed-point property, is here refuted by a rather simple example. In addition, it is shown that a certain cone, namely the join of the above example with a point, does not have the fixed-point property. *E. G. Begle.*

Strother, Wayman L. On an open question concerning fixed points. Proc. Amer. Math. Soc. **4**, 988–993 (1953).

Continuous point-to-set mappings of a space X into itself are studied. A point x in X is called a fixed point of such a mapping, F, if x is in $F(x)$. The space X is said to have the F.p. property if every continuous point-to-set mapping of X into itself has a fixed point. It is first shown that, while the unit interval I does have the F.p. property, the square, $I\times I$, does not. Next it is shown that two restricted classes of mappings do have fixed points. The first class is that of mappings F which are such that for some Tychonoff cube T there is a homeomorphism h: $X\rightarrow h(X)\subset T$ such that $h(X)$ is a retract of T and, for each x in X, $hF(x)$ is a cartesian product of subsets of the factors of T. The second class of mappings is defined in a similar manner in terms of another special property of the subsets $hF(x)$ of T. In each case, the proof that such a mapping has a fixed point proceeds by constructing a trace of F, i.e., a continuous point-to-point mapping f: $X\rightarrow X$ such that for each x, $f(x)$ is in $F(x)$.
E. G. Begle (New Haven, Conn.).

Vesentini, Edoardo. Sui punti uniti delle trasformazioni topologiche delle superficie orientate. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11**, 224–237 (1952).

A homeomorphism T of a metric space S onto itself is regular at x_0. If the system $\{T^n(x)\}$ is equicontinuous at x_0, assume S to be a surface. The author obtains properties of T assumed to be regular on certain specified continua. For example, let S be a bounded region in E^2 bounded by three disjoint circles, the inside ones being γ_1 and γ_2. Assume that T preserves orientation and leaves each γ invariant. Then T can not be regular on the γ's and on a simple arc in S joining the γ's unless T leaves the γ's pointwise fixed.
P. A. Smith (New York, N. Y.).

Ball, B. J. Some theorems concerning spirals in the plane. Amer. J. Math. **76**, 66–80 (1954).

An arc t in the plane is said to spiral down on a point 0 provided there exists no finite sequence of topological rays from 0, no one of which intersects its predecessor except in 0, such that the first intersects t only in 0 and the last is a straight ray. A point set in the plane which contains no arc which spirals down on a point is said to be spiral-free. An example is given of a compact totally disconnected set such that every arc containing it spirals down on uncountably many points. Results are established leading to the conclusion that for a compact totally disconnected plane set M to be a subset of a spiral-free arc it is sufficient that the projection of M on some straight line be totally disconnected and it is necessary and sufficient that M be contained in the union of a continuous and equicontinuous collection of disjoint spiral free arcs whose hyperspace is totally disconnected. *G. T. Whyburn* (Charlottesville, Va.).

Fuller, F. B. The homotopy theory of coincidences. Ann. of Math. (2) **59**, 219–226 (1954).

Suppose f, g are maps of a complex K into a manifold M. The problem considered is that of deciding when f, g are homotopic respectively to maps f', g' without coincidences. If L is a subcomplex of K containing no coincidences of f and g, the problem may be generalized by asking when f, g may be deformed into maps without coincidences in such a way that no coincidences appear in L throughout the deformations. In this formulation the problem appears naturally as that of measuring the successive obstructions to deforming a map of K, L to $M\times M$, $M\times M-D$ down into $M\times M-D$, where D is the diagonal in $M\times M$. Assuming M simply-connected, the primary obstruction is calculated if $\dim M\geqq 3$; and the secondary obstruction is calculated if $\dim M\geqq 4$, using the Steenrod technique of functional cup products [Ann. of Math. (2) **50**, 954–988 (1949); these Rev. **11**, 122]. Thus algebraic criteria are furnished for deciding whether two maps of an $(n+1)$-dimensional complex into a simply-connected n-manifold $(n\geqq 4)$ can be freed of coincidences. *P. J. Hilton.*

Whitehead, J. H. C. On certain theorems of G. W. Whitehead. Ann. of Math. (2) **58**, 418–428 (1953).

The author corrects the sign in three theorems of G. W. Whitehead [Ann. of Math. (2) **43**, 634–640 (1942), Theorem 2; ibid. **47**, 460–475 (1946), Theorem (3.2); ibid. **51**, 192–237 (1950), Theorem (5.1); these Rev. **4**, 88; **8**, 50; **12**, 847]. Each theorem asserts an equality, and the corrected versions replace one side of the equality by its negative. The proofs involve careful attention to orientation, and the corrected theorems are used in a later paper (see the following review).
E. Spanier (Chicago, Ill.).

Hilton, P. J., and Whitehead, J. H. C. Note on the Whitehead product. Ann. of Math. (2) **58**, 429–442 (1953).

The authors study certain operations on the homotopy groups of spheres. In particular, if $[\alpha, \beta]\in\pi_{p+q-1}(S^n)$ denotes the Whitehead product of $\alpha\in\pi_p(S^n)$, $\beta\in\pi_q(S^n)$, then the homomorphism P: $\pi_p(S^n)\rightarrow\pi_{p+n-1}(S^n)$ defined by $P(\alpha)=[\alpha, \iota_n]$ where ι_n is a generator of $\pi_n(S^n)$ is studied in some detail. It has been proved (see the following review) that if $\beta\in E\pi_{q-1}(S^{n-1})$, where E denotes the suspension homomorphism, then $[\alpha, \beta]=\pm P(\alpha\circ E^{p-n}\beta)$ which shows that the homorphism P plays an important role in a general consideration of the Whitehead product.

The homomorphism P is shown to be related to the existence of vector fields on spheres. For example, if S^n admits m vector fields linearly independent at every point ($m \geq 1$) then $P(\alpha)$ is an m-fold suspension for every $\alpha \in \pi_p(S^n)$ where $p \geq 1$. Also if S^{p+1} admits ($p-n$) vector fields linearly independent at every point but every family of ($p-n+1$) vector fields is somewhere dependent (where $0<n<p<2n-1$), then $P\alpha=0$ for some non-zero $\alpha \in \pi_p(S^n)$. *E. Spanier.*

Barratt, M. G., and Hilton, P. J. On join operations in homotopy groups. Proc. London Math. Soc. (3) 3, 430–445 (1953).

The main result of this paper is a characterization of the "join" operation [for the definition, see G. W. Whitehead, Ann. of Math. (2) 51, 192–237 (1950); these Rev. 12, 847] when applied to homotopy groups of spheres in terms of the Freudenthal suspension and the operation of composition. The author also introduces a new operation on elements of homotopy groups of spheres; this new characterization of the join operation together with the new operation enables him to compute many Whitehead products. Application is also made to the calculation of the generalized Hopf homomorphism $H^*: \pi_r(S^n) \to \pi_{r+1}(S^{2n})$. *W. S. Massey.*

Bott, R., and Samelson, H. On the Pontryagin product in spaces of paths. Comment. Math. Helv. 27 (1953), 320–337 (1954).

Let E be the space of paths in a 0-connected space X that end at x_0, and Ω the loops at x_0; E is a fiber space over X with fiber Ω and projection sending each path to its initial point. Ω admits the usual multiplication, which therefore induces a multiplication in $H(\Omega)$ (pairing $H_m(\Omega)$, $H_n(\Omega)$ to $H_{m+n}(\Omega)$) and makes $H(\Omega)$ into a ring having the zero homology class x_0 as unit. This multiplication can be regarded as that induced by the map $\gamma: E\times\Omega \to E$ (when restricted to $\Omega\times\Omega$) which sends (e, ω) to the path $e\omega$. Using the spectral sequence $E_r = \sum E_r^{p,q}$ of the fiber space E, the authors show that if X is 1-connected, γ also induces pairings $E_r^{p,q}$, $H_n(\Omega)$ to $E_r^{p,q+n}$ ($r \geq 1$) and $E_0^{p,q}$, $C_n(\Omega)$ to $E_0^{p,q+n}$ which are bilinear, associative and commute with the identification $E_{r+1}=H(E_r)$. A general theorem on the structure of the ring $H(\Omega)$ follows from this which contains, among others, the assertion: if X is the union of k spheres of dimensions $n_i>1$ having exactly one point in common, then $H(\Omega)$ is the free associative algebra on k generators of dimensions n_i-1. *J. Dugundji* (Los Angeles, Calif.).

Serre, Jean-Pierre. Cohomologie modulo 2 des complexes d'Eilenberg-MacLane. Comment. Math. Helv. 27, 198–232 (1953).

The main purpose of this paper is to give complete explanations and proofs of some results announced without proof previously [C. R. Acad. Sci. Paris 234, 1243–1245 (1952); these Rev. 13, 675]. The main result obtained is a complete determination of the cohomology algebra with coefficients modulo two of the Eilenberg-MacLane complexes $K(\Pi, n)$ in the case where Π is a finitely generated abelian group. It should be pointed out that results of §3 of the present paper were not announced in the note cited above. In this § the author studies the asymptotic behavior of the Poincaré series (the natural generalization of the Poincaré polynomial) of the cohomology algebra (modulo two) of the Eilenberg-MacLane complexes. By using methods similar to those of analytic number theory, he is able to prove that if X is a topological space satisfying certain very general conditions (for example, if X is a simply connected finite polyhedron having non-trivial homology modulo two) then there exists an infinitude of dimensions n such that the homotopy group $\pi_n(X)$ contains an element of order two or of infinite order. *W. S. Massey.*

***Thom, R. Sur les variétés cobordantes.** Colloque de topologie et géométrie différentielle, Strasbourg, 1952, no. 7, 4 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1953.

The main purpose of this note is to complete and correct some results announced previously by the author [Colloque de Topologie de Strasbourg, 1951, no. V, Univ. de Strasbourg, 1952; these Rev. 14, 492]. It should be pointed out that since the date of this colloquium, the author has obtained many additional results on cobounding manifolds [cf. C. R. Acad. Sci. Paris 236, 1733–1735 (1953); these Rev. 14, 1112]. *W. S. Massey* (Providence, R. I.).

***Thom, M. R. Variétés différentiables cobordantes.** Géométrie différentielle. Colloques Internationaux de Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 143–149. Centre National de la Recherche Scientifique, Paris, 1953.

The results discussed in this colloquium address have been already published (without proof) by the author [cf. C. R. Acad. Sci. Paris 236, 1733–1735 (1953); these Rev. 14, 1112]. *W. S. Massey* (Providence, R. I.).

Massey, W. S. Some new algebraic methods in topology. Bull. Amer. Math. Soc. 60, 111–123 (1954).

An expository account of some of the new algebraic machinery used in algebraic topology with particular emphasis on spectral sequences and exact couples. Some definitions and applications are given but most of the details are omitted. *E. H. Spanier* (Chicago, Ill.).

GEOMETRY

Lorent, H. Sur la section de l'angle. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 39, 1027–1040 (1953).

Reeb, O. Zur Dimension des Winkels. Optik 11, 75–94 (1954).

Piza, Pedro A. Une généralisation du théorème de Pythagore. Mathesis 63, 26–28 (1954).

Boomstra, W. Inequalities in the triangle. Nieuw Tijdschr. Wiskunde 41, 197–202 (1954). (Dutch)

Thébault, Victor. Au sujet de l'orthopôle. Mathesis 63, 21–26 (1954).

Court, Nathan Altshiller. Sur les tétraèdres circonscrits par les arêtes à une quadrique. Mathesis 63, 12–18 (1954).

Petrovitch, M. Sur les inégalités stéréometriques. Srpska Akad. Nauka. Zbornik Radova 35. Mat. Inst. 3, 1–4 (1953). (Serbo-Croatian. French summary)

Jovičić, Milorad M. Unmittelbare graphische Restitution in der schiefen Axonometrie. Srpska Akad. Nauka. Zbornik Radova **35**. Mat. Inst. **3**, 153–156 (1953). (Serbo-Croatian. German summary)

Cavallaro, Vincenzo G. Sur l'emploi des axes de l'ellipse de Steiner. Mathesis **63**, 29–36 (1954).

Devidé, Vladimir. Ein Satz über homothetische Hyperellipsoide im n-dimensionalen Raume. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **8**, 194–195 (1953). (Serbo-Croatian. German summary)

***Hosemann, R. Verfolgungskurven.** Naturforschung und Medizin in Deutschland, 1939–1946, Band 3. Angewandte Mathematik, Teil I, pp. 269–307. Verlag Chemie, Weinheim, 1953. DM 20.00.

Rangaswami Aiyer, K. On a quartic curve associated with a tetrahedron. Math. Student **21** (1953), 87–96 (1954).

The author studies the biquadratic curve sometimes referred to as the Schroeter quartic. Using synthetic methods he derives a considerable number of properties of the curve. Most of those, however, have been anticipated by earlier writers on this topic. [See A. Marmion, Mathesis **58**, 30–43 (1949) [these Rev. **11**, 125] and the bibliographical references given in connection with that paper.]

N. A. Court (Norman, Okla.).

***Rehbock, F. Darstellende und konstruktive Geometrie.** Naturforschung und Medizin in Deutschland, 1939–1946, Band 3. Angewandte Mathematik, Teil I, pp. 259–261. Verlag Chemie, Weinheim, 1953. DM 20.00.

***Forder, H. G. Coordinates in geometry.** Auckland University College, Auckland, 1953. 32 pp.

There are three chapters in this monograph. In the first there is an elegant presentation of the consequences of the minor theorem of Desargues and its equivalence to coordinates from an alternative division ring. This leads naturally to the full theorem of Desargues and associative coordinates. The second chapter, without axioms of order deals with congruence of angles (crosses) and line segments (point pairs) in a Pappian geometry. The third chapter uses properties of convex functions to develop the trigonometry of the hyperbolic plane. *Marshall Hall, Jr.*

Have

***Prüfer, Heinz. Projektive Geometrie.** 2te Aufl. Akademische Verlagsgesellschaft, Geest und Portig K.-G., Leipzig, 1953. vii+314 pp. DM 9.00.

The 1st edition appeared in 1939 [cf. these Rev. **1**, 80]. Only minor changes have been made in this edition.

Cesarec, Rudolf. Über die Inzidenzgleichung in nichtzusammengehörigen projektiven Koordinaten. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **8**, 168–174 (1953). (Serbo-Croatian. German summary)

Sosnowski, W. Sur une interprétation géométrique des éléments complexes. Ann. Soc. Polon. Math. **24** (1951), no. 2, 35–48 (1954).

Von Staudt [Beiträge zur Geometrie der Lage, Heft I, Bauer and Raspe, Nürnberg, 1856] represented each pair of conjugate complex points on a line by the elliptic involution that has these for its invariant points, and distinguished the two by the apparently artificial device of associating each with one of the senses along the line. The author uses, instead, a projectivity of period 3 and its inverse; there is then just one imaginary point for each such projectivity. He uses the symbol $)x\,y\,z($ to denote the imaginary point represented by projectivity $x\,y\,z \barwedge y\,z\,x$, where x, y, z are three real points on the line. Restricting consideration to two dimensions, he gives the dual representation for an imaginary line, and shows that the line joining $)x\,y\,z($ and $)x\,u\,v($ is $)tx,\ ty,\ tz(=)tx,\ tu,\ tv($, where t is the point of intersection of the two real lines yu and zv. The construction for the intersections of a conic with an exterior line is given in an affine form, for convenience. Making a brief excursion into Euclidean geometry, he remarks that the isotropic lines through any point are represented by the rotations through $\pm 120°$. He shows also that projectivities of periods 4 or 6 can be used just as well as those of period 3; in fact, the construction for the intersections of a line and a conic is particularly simple in the case of period 4.

H. S. M. Coxeter (Toronto, Ont.).

Weitzenböck, Roland. Zum Transversalenproblem. II. Zum 9-Ebenenproblem im R_5. Monatsh. Math. **57**, 265–306 (1954).

A set of 9 planes in R_5 has 42 transversal planes [Weitzenböck, Monatsh. Math. **57**, 185–198 (1953); these Rev. **15**, 467]. The problem at hand is to reduce the finding of these planes to the solution of algebraic equations and to give the corresponding constructions in the projective geometry. The problem is not completely finished here, but the author does give a computation which leads to many interesting figures. The method used is briefly this: Let the nine given planes be called $1^3, 2^3, \cdots, 9^3$. Fix point A in 1^3 and point B in 2^3. In general there is one plane through A and B which cuts $3^3, 4^3, 5^3$. Ask for those special points B in 2^3 for which the plane through A and B will cut 6^3 in addition to $3^3, 4^3, 5^3$. In general, these points B in 2^3 lie on a curve of degree 3, $K_{3456}=0$. There are also in 2^3 the curves $K_{3457}=0$ and $K_{3458}=0$. In order that there shall be a $T_{12345678}$ through the point A in 1^3, i.e. a transversal for the planes $1^3, \cdots, 8^3$, it is necessary that the three $K=0$ curves intersect in a point B^*. The simultaneous solution of these equations is reduced to linear problems, and the author finds a condition $\Delta_\lambda^{678}(A)=0$ which a point A in 1^3 is to satisfy if there is to be a $T_{12345678}$ through A. There is also in 1^3 the curve $\Delta_\lambda^{679}(A)=0$, on which lie the points A through which it is possible to construct a $T_{12345679}$. These curves are in general of order 135. The points of 1^3 through which a transversal $T_{123456789}$ can be constructed will be included among the intersections of $\Delta_\lambda^{678}(A)=0$ and $\Delta_\lambda^{679}(A)=0$. There are special cases in several proofs, and each one is investigated.

A. Schwartz (New York, N. Y.).

Fladt, Kuno. Über die Transformationen der Hauptgruppe in der nichteuklidischen Geometrie und die komplexe Trigonometrie. J. Reine Angew. Math. **192**, 129–154 (1953).

Extending to non-Euclidean spaces his work on Euclidean transformations [Math.-Phys. Semesterber. **2**, 104–116 (1951); these Rev. **12**, 849], the author uses both quaternions and biquaternions. So far as elliptic space is concerned, the results somewhat resemble the reviewer's [Amer. Math. Monthly **53**, 136–146 (1946); these Rev. **7**, 387]. But, by introducing a parameter k into the definition of his biquaternions, the author is able to deal with congruent transformations in hyperbolic space ($k<0$) as well as in Euclidean space ($k=0$) and elliptic space ($k>0$). An im-

portant by-product is the trigonometry of the general rectangular skew hexagon. *H. S. M. Coxeter.*

Szász, Paul. Über die Hilbertsche Begründung der hyperbolischen Geometrie. Acta Math. Acad. Sci. Hungar. **4**, 243–250 (1953). (Russian summary)

Szász, Paul. Herleitung der hyperbolischen Trigonometrie in der Poincaréschen Halbebene. Acta Sci. Math. Szeged **15**, 126–129 (1954).

In the first paper the author gives a one to one mapping of the hyperbolic plane onto a Euclidean half-plane. This mapping is used to derive hyperbolic geometry without using an axiom of continuity.

In the second paper hyperbolic plane trigonometry is derived by means of this mapping. The method is very similar to the one used by Eves and Hoggatt [Amer. Math. Monthly **58**, 469–474 (1951); these Rev. **13**, 269].

E. Lukacs (Washington, D. C.).

Szász, Paul. Über die Rektifikation des Kreises, des Grenzkreises und der Abstandslinie. Acta Math. Acad. Sci. Hungar. **4**, 251–253 (1953). (Russian summary)

Using previous results on hyperbolic geometry [Acta Sci. Math. Szeged **12**, Pars A, 44–52 (1950); these Rev. **12**, 276], the author shows by elementary methods that the ratio of the length of an arc of a circle (or horocycle or equidistant curve) to its chord tends to one as the length of the arc tends to zero. *E. Lukacs* (Washington, D. C.).

Belyaev, M. G. The tractrix and pseudosphere in Lobačevskiĭ space. Dopovidi Akad. Nauk Ukrain. RSR 1951, 312–319 (1951). (Ukrainian. Russian summary)

Algebraic Geometry

Hutcherson, W. R., et Childress, N. A. Etude d'une involution cyclique de période cinq. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 103–108 (1954).

Peeples, W. D., Jr. Elliptic curves and rational distance sets. Proc. Amer. Math. Soc. **5**, 29–33 (1954).

By using the method of tangentials and some simple arguments of valuation theory, conditions are obtained sufficient to ensure that a rational cubic curve

$$ax(y^2-1)-by(x^2-1)=0$$

will contain an infinite number of rational points. This occurs if, for example, $x=m/n$, $y=p/q$ is a point of the curve, where m, n, p, q are non-zero integers, of which m, p are even and n, q are odd. Correspondingly, there is an infinite number of points $(u, 0)$, where $u=2by/(y^2-1)$ and (x, y) is a rational point of the cubic curve, which together with $(0, \pm a)$ and $(0, \pm b)$ form a set in which all distances are rational. *B. Segre* (Rome).

Piazzolla Beloch, M. Alcune osservazioni sulla simmetria obliqua nelle curve algebriche piane. Ann. Univ. Ferrara. Sez. VII. (N.S.) **2**, 151–154 (1953).

It is stated as obvious (with justice) that the necessary and sufficient condition for a line $\alpha x+\beta y+\gamma=0$ to be an axis of symmetry (orthogonal or oblique with direction $y:x=l:m$) of the curve $f(x, y)=0$ (of order n), is that $\alpha x+\beta y+\gamma$ is a divisor of $l\partial f/\partial x+m\partial f/\partial y$, and that the remaining factor (of order $n-2$) has the same line as axis of symmetry with the same directions. Thus by a method of repeated division it is possible to find the axes of symmetry of any curve; the method is applied to the curve $x^4/a^4+y^4/b^4=1$, with the expected results. *P. Du Val.*

Godeaux, Lucien. Recherches sur les points de diramation de troisième catégorie d'une surface multiple. I, II, III, IV, V. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 1013–1023, 1087–1093 (1953); **40**, 81–86, 200–208, 355–370 (1954).

Hall, Raymond. Some types of irregular threefolds. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11**, 167–175 (1952).

There are given here two examples of irregular threefolds, which are primals in the four-dimensional projective space S_4. The first type of threefold is obtained by equating to zero a general form of degree r in A_1, A_2, A_3, with constant coefficients, where A_1, A_2, A_3 are general primals in S_4 of order m. The other type has the equation:

$$P^2(U^2+V^2)+Q^2(U^2-V^2)=0,$$

where P and Q are general forms of degree m and U, V are general forms of degree n. This last type of equation has been employed in S_3 by Pedoe [Proc. Cambridge Philos. Soc. **31**, 48–49 (1935)]. *P. Abellanas* (Madrid).

Caputo, Michele. Sugli spazi totali dei sistemi algebrici di spazi. Ann. Univ. Ferrara. Sez. VII. (N.S.) **2**, 45–52 (1953).

This paper computes the dimension of the system of total spaces of dimension t which belong to the system of h-spaces in S_r defined by the vanishing of m generic forms in the Grassmann coordinates. The method is based on the principle of counting constants, and the reasoning depends on a number of assertions concerning the completeness of linear systems which are not obvious to the reviewer.

W. V. D. Hodge (Cambridge, England).

Segre, Beniamino. Sulla totalità delle varietà algebriche razionali di uno spazio complesso. Boll. Un. Mat. Ital. (3) **8**, 374–377 (1953).

The author calls an algebraic variety V (in the complex projective space S_n) rational, resp. real, if V can be defined by equations with rational, resp. real, coefficients. Let V_p^* be a pure p-dimensional rational variety in S_n $(0\leq p\leq n-1)$. The following is proved: in any neighborhood of the identical transformation there exists a projective transformation T of S_n such that the T-transform of V is a real variety which does not meet (in the complex S_n) any rational variety of dimension $<n-p$. The proof of this result is based on the following lemma: if P is a real simple point of a rational irreducible variety V in S_n, then any neighborhood of P on V contains a real point O with the property that any rational variety which contains O contains the whole variety V (in other words: O is a general point of V over the field of rational numbers). *O. Zariski.*

Nakai, Yoshikazu. On the independency of differential forms on algebraic varieties. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **28**, 67–80 (1953).

Démonstration algébrique du théorème suivant: si des formes différentielles (ω_i) sur une variété projective V sont linéairement indépendantes, elles induisent des formes linéairement indépendantes sur tout section de V par une

hypersurface générique de degré assez grand; dans le cas classique un théorème analogue avait été démontré par Igusa [Amer. J. Math. 74, 1–22 (1952); ces Rev. 13, 680] pour les sections de V par des hypersurfaces génériques de tous degrés, mais seulement pour des formes de première espèce. L'auteur démontre d'abord divers lemmes relatifs aux paramètres uniformisants. Puis il montre que, si les (ω_i) induisent des formes linéairement indépendantes sur la section de V par une hypersurface H de degré m, alors elles induisent des formes linéairement indépendantes sur toute section de V par une hypersurface générique de degré $\geqq m$. Enfin il construit une telle hypersurface H, en distinguant divers cas dûs à la dimension de V et au degré des ω_i.

P. Samuel (Clermont-Ferrand).

Differential Geometry

Forte, Bruno. Di alcune proprietà cinematiche riguardanti il moto rigido di una sfera su se stessa. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 14, 765–770 (1953).

Considering the motion of a sphere in itself at a given moment the author generalizes the concept of the "cammino cinematico" as introduced by Sobrero [same Rend. (8) 8, 360–364 (1950); these Rev. 12, 364]. Two points P and P' of the fixed and the moving sphere respectively may be conjugated points of the nth order. For $n=2$ it is sufficient for P and P' to be on the same great circle through the instantaneous centre, for $n=3$ they must satisfy the Savary formula; the locus of conjugate points for $n=4$ is a spherical cubic.

O. Bottema (Delft).

Godeaux, Lucien. Sur les congruences engendrées par les directrices de Wilczynski d'une surface. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 40, 209–218 (1954).

Etude des congruences engendrées par les directrices de Wilczynski d'une surface (x) dans l'hypothèse où les plans focaux de la directrice passant par le point x, passent par les foyers de la seconde directrice.

Résumé de l'auteur.

Godeaux, Lucien. Remarque sur les suites de Laplace inscrites dans une suite de Laplace. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 40, 87–90 (1954).

Geĭdel'man, R. M. Stratification of two-parameter families of straight lines in a multidimensional projective space. Doklady Akad. Nauk SSSR (N.S.) 93, 957–960 (1953). (Russian)

In an n-dimensional projective space P_n two two-parameter families of straight lines (l_1) and (l_2) being given, (l_1) stratifies (l_2) if there exists a one-parameter family of surfaces (Σ) such that the tangent planes at points of intersection of l_1 and Σ all pass through l_2. If (l_2) also stratifies (l_1), the two families are completely stratified. It is first shown that if (l_1) stratifies (l_2) then (l_1) is a congruence and then it follows that if (l_1) and (l_2) are completely stratified they are congruences in P_3 imbedded in P_n. The author obtains a number of results in case of stratification in one direction dealing with the existence of conjugate nets and asymptotic lines for parabolic congruences.

M. S. Knebelman (Pullman, Wash.).

Demaria, Davide Carlo. I sistemi di superficie con la proprietà proiettiva o conforme in prima approssimazione. Boll. Un. Mat. Ital. (3) 8, 409–413 (1953).

The author considers the system of ∞^4 surfaces, $z=z(x, y)$, defined by a completely integrable system of analytic partial differential equations of the form

$$z_{xx}=f(x, y, z, z_x, z_y, z_{xy}), \quad z_{yy}=g(x, y, z, z_x, z_y, z_{xy}).$$

If P_1: (x_1, y_1, z_1), P_2: (x_2, y_2, z_2) are neighboring points, ∞^2 of the surfaces pass through both points. The condition that the tangent planes of these ∞^2 surfaces at P_1 be projectively related to the tangent planes at P_2, for every pair of neighboring points P_1 and P_2, can be expressed by the identical vanishing of four expressions Φ_i. In general, if the Φ_i are expanded in powers of x_2-x_1 and y_2-y_1, the terms of lowest degree appearing are of degree 9. If these terms vanish, the system of surfaces is said to possess the projective property in the first approximation. It is shown that a necessary and sufficient condition for the system of surfaces to possess the projective property in the first approximation is that the defining system of differential equations be of the special form

$$z_{xx}=Az_{xy}+B, \quad z_{yy}=Cz_{xy}+D,$$

where A, B, C, D are functions of x, y, z, z_x, z_y.

The condition that the family of ∞^2 tangent planes at P_1 be conformally related to the family of tangent planes at P_2 is also considered. It is found that a necessary and sufficient condition for the system of surfaces to possess the conformal property in an appropriately defined first approximation is that the system of differential equations be of the form

$$z_xz_yz_{xx}=(1+z_x^2)z_{xy}+Bz_xz_y, \quad z_xz_yz_{yy}=(1+z_y^2)z_{xy}+Dz_xz_y.$$

L. A. MacColl (New York, N. Y.).

Abramov, A. A. A formula of Gauss-Bonnet type for the tensor fields of Pontryagin. Doklady Akad. Nauk SSSR (N.S.) 93, 757–758 (1953). (Russian)

Let A_n be an n-dimensional affinely connected manifold containing a region D_r on whose boundary, C_{r-1}, there is defined a field E of n-frames. In an earlier paper [Uspehi Matem. Nauk (N.S.) 5, no. 2(36), 162–163 (1950); these Rev. 12, 131] the author showed that to every r-form Φ on D_r there is a form X such that $\int_{D_r}\Phi-\int_{C_{r-1}}X$ is invariant under continuous (infinitesimal) deformations of the connection or of the field E. In the present work the form X is computed in case Φ is a combination of curvature forms. It is pointed out that, if the connection is induced by a Riemannian metric in the neighborhood of the cycle, a previous theorem of the author's [Doklady Akad. Nauk SSSR (N.S.) 81, 125–128 (1951); these Rev. 13, 869] applies to show that all forms invariant in this sense whose components are analytic functions of the connection and its derivatives are polynomials in the forms belonging to the Pontryagin characteristic classes.

L. W. Green.

Takizawa, Seizi. On the Stiefel characteristic classes of a Riemannian manifold. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. 28, 1–10 (1953).

In line with the objective of the Gauss-Bonnet formula the author gives an integral formula for the Stiefel classes of a Riemannian manifold. The differential forms of the integrals belong to appropriate associated bundles, and not to the base space.

S. Chern (Chicago, Ill.).

Takizawa, Seizi. On the primary difference of two frame functions in a Riemannian manifold. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **28**, 11–14 (1953).

[Cf. the preceding review.] The method of the above paper is applied to derive an integral formula for the difference cocycle of two frame functions of a Riemannian manifold. *S. Chern* (Chicago, Ill.).

Yagyu, Toshikazu. On the Whitney characteristic classes of the normal bundle. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **28**, 15–17 (1953).

The author gives an integral formula for the Whitney characteristic classes of the normal bundle of a submanifold of a Riemannian manifold. [Cf. the two preceding reviews.] *S. Chern* (Chicago, Ill.).

Have

***Willmore, Thomas J. Quelques propriétés locales et globales des espaces riemanniens harmoniques.** Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 89–95. Centre National de la Recherche Scientifique, Paris, 1953.

Let Δ_2 be the Laplace operator associated with a Riemannian space V_n. Then V_n is called harmonic if, at each point M_0, the equation $\Delta_2 V=0$ has a non-trivial solution which depends only on the geodesic distance from M_0. The author first states some known characterisations of harmonicity due to A. Lichnerowicz and A. G. Walker, and then uses them to prove some global properties. Among other results, it is shown that (i) the only compact, connected semi-simple Lie group of which the two-sided invariant metric is harmonic is the rotation group of three dimensions (or its universal covering group); (ii) each connected, compact two-point homogeneous space with the natural metric is harmonic. *H. C. Wang.*

Spencer, D. C. A generalization of a theorem of Hodge. Proc. Nat. Acad. Sci. U. S. A. **38**, 533–534 (1952).

Soit M une variété de Riemann réelle, de dimension n et de classe C^∞. On dit que M est uniformément régulière (smooth) si: (i) il existe un nombre positif η tel que deux points quelconques de M dont la distance n'excède pas η, peuvent être joints par une géodésique unique; (ii) dans la sphère géodésique de rayon ϵ, centrée en $p \in M$, les dérivées partielles d'ordre $\leqq 4$, du tenseur métrique, par rapport aux coordonnées normales en p, sont bornées par un nombre indépendant de p. Si la variété M est uniformément régulière et complète pour la métrique, on a les deux théorèmes suivants. (A) Toute forme différentielle φ de degré q, $0 \leqq q \leqq n$, de norme finie, qui satisfait à $\Delta\varphi=0$, satisfait aussi à $d\varphi=\delta\varphi=0$. (B) (généralisation du théorème de Hodge) Etant donnée une q-forme fermée α, de norme finie, il existe une forme harmonique unique, de norme finie, ayant les mêmes périodes que α sur tous les q-cycles à supports compacts. L'auteur annonce une démonstration basée sur la méthode de l'équation de la chaleur introduite par Milgram et Rosenbloom [Proc. Nat. Acad. Sci. U. S. A. **37**, 180–184 (1951); ces Rev. **13**, 160]. *P. Dolbeault.*

Otsuki, Tominosuke. On the existence of solutions of a system of quadratic equations and its geometrical application. Proc. Japan Acad. **29**, 99–100 (1953).

The author proves the following theorem, conjectured by the reviewer and N. H. Kuiper [Ann. of Math. (2) **56**, 422–430 (1952); these Rev. **14**, 408]: Let

$$\psi_\alpha(x)\equiv\sum_{i,j}A_{\alpha ij}x^ix^j=0,\quad A_{\alpha ij}=A_{\alpha ji},$$
$$i,j=1,\cdots,n;\ \alpha=1,\cdots,N$$

be a system of quadratic equations in x^i. If

$$\sum_{i,j,h,k=1}^{n}\sum_{\alpha=1}^{N}(A_{\alpha ih}A_{\alpha jk}-A_{\alpha ik}A_{\alpha jh})x^iy^jx^hy^k\leqq 0$$

for any x^i, y^j, the system of quadratic equations has a non-trivial real solution in x^i, when $N<n$. This theorem has the following geometrical consequence: Let M be a compact Riemannian manifold with the property that at every point there is a q-dimensional linear subspace in the tangent space along whose plane elements the sectional curvatures are non-positive. Then M cannot be isometrically imbedded in an Euclidean space of dimension $n+q-1$. *S. Chern.*

Berger, Marcel. Groupes d'holonomie des variétés riemanniennes. Applications. C. R. Acad. Sci. Paris **238**, 985–986 (1954).

After an enumeration of possible restricted homogeneous holonomy groups of non-(Cartan) symmetric, non-locally reducible Riemannian spaces with positive definite metrics [same C. R. **237**, 1306–1308 (1953); these Rev. **15**, 468], the author now deals with the indefinite case of the V_n^h, the equation of whose fundamental quadric $Q^h{}_{n-1}$ can be reduced to h positive and $n-h$ negative squares. Let $\mathrm{SO}^h(n)$ denote the subgroup of $\mathrm{GL}(n, R)$ which leaves $Q^h{}_{n-1}$ invariant, and $\mathrm{SU}^h(n)$, $\mathrm{Sp}^h(n)$, $\mathrm{SO}^*(2n)$ respectively the groups isomorphic to $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, $\mathrm{SO}(2n)$ contained in $\mathrm{SO}^{2h}(2n)$, $\mathrm{SU}^{2h}(2n)$, $\mathrm{SO}^{2n}(4n)$. Let $\mathrm{SO}(n)\times\mathrm{SO}(n)$ denote the real representation of the complex group $\mathrm{SO}(n)$ contained in $\mathrm{SO}^n(2n)$. The result is then, apart from a finite number of exceptional groups:

for $V_m{}^h$: $\mathrm{SO}^h(m)$;
for V_{2n}^{2k} also: $T^1\times\mathrm{SU}^k(n)$, $\mathrm{SU}^k(n)$, and for $V^n{}_{2n}$ also: $\mathrm{SO}(n)\times\mathrm{SO}(n)$;
for V_{4n}^{4k} also: $\mathrm{Sp}(1)\times\mathrm{Sp}^k(n)$, $\mathrm{Sp}^k(n)$; and for V_{4n}^{2n} also: $\mathrm{SO}^*(2n)$.

In the positive-definite case the possible holonomy groups were exactly the closed subgroups of $\mathrm{SO}(n)$ that are transitive on the unit sphere. Here, this is no longer true: the subgroups of $\mathrm{SO}^h(m)$ transitive and effective on $Q^h{}_{m-1}$ are (apart from a finite number of exceptional groups):

for $Q^h{}_{m-1}$: $\mathrm{SO}^h(m)$;
for Q_{2n-1}^{2k} also: $T^1\times\mathrm{SU}^k(n)$, $\mathrm{SU}^k(n)$;
for Q_{4n-1}^{4k} also: $\mathrm{Sp}(1)\times\mathrm{Sp}^k(n)$, $T^1\times\mathrm{Sp}^k(n)$, $\mathrm{Sp}^k(n)$.

Then follows a theorem concerning the possible equality of the restricted and the non-restricted holonomy groups, which shows a connection with Kähler manifolds. Finally, we find a list of possible covariant constant exterior forms. [The three lists in the paper appear to contain some errors. The two lists here in the review have been changed in accordance with corrections supplied to the reviewer by the author.] *A. Nijenhuis* (Princeton).

Karpelevič, F. I. Surfaces of transitivity of a semisimple subgroup of the group of motions of a symmetric space. Doklady Akad. Nauk SSSR (N.S.) **93**, 401–404 (1953). (Russian)

This work is based on the well known results of E. Cartan on semi-simple groups. If $\mathfrak{M}$ is a symmetric Riemann space of negative curvature, its group of motions $\mathfrak{G}$ is semi-simple

and the stationary subgroup $\mathfrak{H}$ is a maximal compact subgroup of $\mathfrak{G}$. Let G be the Lie algebra of $\mathfrak{G}$ and $\varphi(g, h)$, $g, h \in G$, the Cartan invariant bilinear form. Let H be a subspace of G. The set of elements X of G such that $\varphi(x, h)=0$ for all $h \in H$ is called the orthogonal complement of H (in G). Let $\bar{\mathfrak{G}}$ be a semi-simple subgroup of $\mathfrak{G}$ and $\bar{\mathfrak{H}}$ a maximal compact subgroup of $\bar{\mathfrak{G}}$. Let $\bar{G}$ and $\bar{H}$ be their subalgebras and $\bar{X}$ the orthogonal complement of $\bar{H}$ in $\bar{G}$. Then $\bar{G}$ is canonically imbedded in G if there exists a maximal compact subalgebra H of G such that $\bar{H}\subset H$ and $\bar{X}\subset X$. The two theorems the author proves are as follows. Let $\bar{G}$ be canonically imbedded in G and let $\bar{H}\subset H$ and $\bar{X}\subset X$. If M is a point whose stationary subalgebra is H and $\mathfrak{S}$ is the surface of transitivity of $\bar{G}$, containing M, then $\mathfrak{S}$ is totally geodesic (with respect to the metric $\varphi(\alpha, k)$). The other theorem states that if $\bar{\mathfrak{G}}$ is a semi-simple subgroup of $\mathfrak{G}$ then it is canonically imbedded in $\mathfrak{G}$.

M. S. Knebelman.

Ingraham, Richard L. The geometry of the linear partial differential equation of the second order. Amer. J. Math. 75, 691–698 (1953).

It is proved for $n=2$: 1) an equation of the form $g^{rs}\partial^2_{rs}\phi+v^r\partial_r\phi=0$ can always be written in the form $\nabla_r g^{rs}\partial_s\phi=0$ where ∇_r symbolizes the covariant differentiation in a W_n, and the connexion of this W_n is uniquely determined; 2) the equation is self-adjoint if its W_n is a V_n and it is equivalent to an ordinary Laplacian equation if its W_n is an R_n; 3) two equations are equivalent under coordinate transformations and gauge transformations if their Weyl geometries are the same. Necessary and sufficient conditions for this latter equivalence are derived in the usual way; they contain the curvature tensor and its covariant derivatives up to a certain order. The case $n=2$ is discussed in the last section.

J. A. Schouten (Epe).

★Bompiani, E. Sur la théorie des connexions. Colloque de topologie et géométrie différentielle, Strasbourg, 1952, no. 1, 4 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1953.

Whenever a manifold is provided with two linear connections there arise invariants expressing the "deviation" between these two. Some of these invariants are established, and their geometrical properties are discussed.

A. Nijenhuis (Princeton, N. J.).

Bompiani, Enrico. Connessioni affini e geometria riemanniana. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 10, 391–405 (1951).

Bompiani, Enrico. Connessioni affini e geometria riemanniana. Boll. Un. Mat. Ital. (3) 8, 363–368 (1953).

Starting from a manifold with a given linear connection, and a tensor $g_{ik}=g_{ki}$ at a given point x it is possible to define "infinitesimal tensor fields" g_{ik} (at x coinciding with the given g_{ik}) that might be considered as "approximating Riemannian structures." The author constructs such fields, and points out some essential correspondences and differences, compared with the given structure. The first paper mainly deals with symmetric connections, while the second note investigates the role played by the torsion of nonsymmetric connections.

A. Nijenhuis.

★Kuiper, N. H. Sur les surfaces localement affines. Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 79–87. Centre National de la Recherche Scientifique, Paris, 1953.

Have

Ehresmann [Enseignement Math. 35, 317–333 (1936)] has given the following definition of a locally homogeneous manifold X: let X be covered by open sets U_α each with a homeomorphism f_α onto an open set V_α in a homogeneous space H; then X is locally homogeneous if, whenever U_α and U_β overlap and W is a component of this intersection, the map $f_\beta f_\alpha^{-1}$ from $f_\alpha(W)\subset H$ onto $f_\beta(W)\subset H$ is given by a motion of H. The best known examples are the locally euclidean, spherical, and hyperbolic space forms of Clifford-Klein. Previously the author has studied locally conformal spaces [Ann. of Math. (2) 52, 478–490 (1950); these Rev. 12, 283] and locally simili-euclidean spaces [Nederl. Akad. Wetensch., Proc. 53, 1178–1185 (1950); these Rev. 12, 519].

In the present paper locally affine surfaces are studied, i.e., H is the affine plane. The universal covering space $\tilde{X}$ of X is again locally affine and X is said to be convex if $\tilde{X}$ is equal to a convex open subset of H, to be normal if this subset is H itself. By a careful study of the group of covering transformations of $\tilde{X}$ (isomorphic to the fundamental group of X), the author deduces that (1) if X is non-compact and normal, it is a cylinder or a Möbius band and (2) if X is compact and convex, it is a torus or a Klein bottle and $\tilde{X}$ is the affine plane, half-plane or quarter-plane. The torus and Klein bottle also carry unique non-convex affine structures. Detailed discussion and subclassification of these cases is included. It is asserted that the sphere cannot carry a locally affine structure and the question is raised as to whether the torus and Klein bottle are the only closed surfaces which can.

W. M. Boothby (Evanston, Ill.).

Castoldi, Luigi. Calcolo della connessione affine integrale associata ad un punto di una varietà a connessione affine. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 14, 760–765 (1953).

This paper calculates for $1<n$ the nth derivatives of the integral affine connection [E. Bompiani, Ann. Mat. Pura Appl. (4) 24, 257–282 (1945); these Rev. 9, 158].

J. M. Thomas (Durham, N. C.).

★Souriau, Jean-Marie. Géométrie symplectique différentielle. Applications. Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 53–59. Centre National de la Recherche Scientifique, Paris, 1953.

Have

The author gives the basic essentials of symplectic geometry. He starts by defining such notions as unitary affinors, fundamental rotation of a space, and, in terms of this rotation, he gives the difference between Euclidean and symplectic geometry. In terms of the notion of the scalar product of two vectors he defines orthogonality and isotropy (as auto-orthogonality). From this follows the notion of manifolds which are completely isotropic, and, if they have the maximum dimension, these are called saturated isotropic manifolds. The expression $\sum p_i dq_i$ will be a total differential only on a completely isotropic manifold upon which there will therefore exist a scalar function α such that $d\alpha=\sum p_i dq_i$. The author goes on to study surfaces in symplectic space, isotropic manifolds upon a surface, and applicability of surfaces. He ends with applications to the theory of partial differential equations of the first order and the calculus of variations.

E. T. Davies (Southampton).

***Eckmann, Beno. Sur les structures complexes et presque complexes.** Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 151–159. Centre National de la Recherche Scientifique, Paris, 1953.

The first part of the paper is concerned with a discussion of local properties of an almost complex structure as related to the theory of connections. To an almost complex structure J there exists an affine connection Λ, not necessarily symmetric, relative to which J is parallel. In order that a symmetric Λ exists, it is necessary and sufficient that the almost complex structure be without torsion. In order that Λ be Riemannian, it is necessary and sufficient that J be without torsion and admit a Kähler metric. An equivalent condition is given, for an almost complex structure to be without torsion, which is in terms of vector fields and their commutator operators. All these discussions are carried out in the real formulation.

In the second part of the paper a summary is given of recent results of E. Calabi and the author [Ann. of Math. (2) **58**, 494–500 (1953); these Rev. **15**, 244] on properties of complex non-Kählerian structures on the product of a $(2p+1)$- and a $(2q+1)$-dimensional sphere $(q>0)$.

S. Chern (Chicago, Ill.).

Hodge, W. V. D. Structure problems for complex manifolds. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11**, 101–110 (1952).

Lecture given in March 1952.

Apte, Madhumalati. Sur certaines variétés hermitiques. C. R. Acad. Sci. Paris **238**, 1091–1093 (1954).

Let $F_{\alpha\beta^*}$ be the skew-symmetric hermitian tensor associated with the fundamental hermitian tensor $g_{\alpha\beta^*}$ in the usual fashion. The metric is called a δF-metric if $\delta F=0$, where

$$(\delta F)_\alpha=g^{\rho\beta^*}\nabla_\rho F_{\alpha\beta^*}+g^{\rho^*\beta}\nabla_{\rho^*}F_{\alpha\beta}=g^{\beta\rho^*}(dF)_{\alpha\beta\rho^*}$$

(this does not imply that the condition $\alpha F=0$ for a Kähler manifold is satisfied). It is shown that for a δF-metric the complex analytic coordinates are isothermic $[\Delta(z^\alpha)=0]$, and conversely. Also, a fairly simple expression is found for the Ricci tensor of a δF-manifold. *A. Nijenhuis.*

***Dedecker, Paul. Les systèmes d'équations extérieures.** Colloque de topologie et géométrie différentielle, Strasbourg, 1952, no. 2, 13 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1953.

The purpose of this article is the study of the integral manifolds of a system S of exterior differential forms in a real-analytic manifold V_n, i.e. submanifolds on which the forms of S reduce to 0. A main theorem describes conditions on an ideal I of the algebra of differential forms, under which a p-dimensional integral manifold of I can be extended (at least locally) to an integral manifold of dimension $p+1$. The first part of the paper treats the necessary algebraic preliminaries: Grassmann algebra over a vector space V; its dual, the Grassmann algebra of forms; ideals in this; notion of a zero of a set S of forms: a subspace of V, on which all forms of S reduce to 0. Theorems of Lepage and Cartan on ideals of forms are given. Certain more refined notions are introduced which are necessary in the study of differential forms: the polar element of a p-subspace E of V consists of all vectors which together with E span a $(p+1)$-zero of S; ordinary and regular p-zeros, characteristic numbers and genus of an ideal of forms, etc. The books by Kähler [Einführung in die Theorie der Systeme von Differentialgleichungen, Teubner, Leipzig-Berlin, 1934] and Schouten and v.d. Kulk [Pfaff's problem and its generalizations, Oxford, 1949; these Rev. **11**, 179] contain similar material. *H. Samelson* (Princeton, N. J.).

Kawaguchi, Akitsugu. Theory of areal spaces. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **12** (1953), 373–386 (1954).

This is an expository article, and is a review of the results obtained by the author and his school in the domain of areal spaces [cf. Kawaguchi, these Rev. **12**, 536; **13**, 384, 385; Kawaguchi and Katsurada, these Rev. **14**, 88; Kawaguchi and Tandai, these Rev. **14**, 586; Tandai, these Rev. **15**, 254].

E. T. Davies (Southampton).

Ide, Saburo. On the connection of a special Kawaguchi space. Tensor (N.S.) **2**, 169–174 (1952).

Unter den Kawaguchischen Räumen, die bekanntlich Mannigfaltigkeiten von Flächenelementen höherer Ordnung sind, hat Kawaguchi selbst denjenigen Spezialfall betrachtet, indem die Metrik durch $s=\int(A_ix''^i+B)^{p^{-1}}dt$ bestimmt ist. Die Forderung der Unabhängigkeit von dem Parameter t führt zur Beziehung (1) $A_i(x,x')x'^i=0$. Dadurch sind nun jedem Raumpunkte zwei Vektoren A_i und x'^i zugeordnet für die (1) gilt. Nun hat W. Wirtinger ganz allgemein eine Übertragung für solche Mannigfaltigkeiten angegeben, bei denen jedem Punkte ein Vektorpaar η^α und v_α mit der Inzidenzbedingung $\eta^\alpha v_\alpha=0$ zugeordnet ist. Die speziellen Kawaguchischen Räume gestatten daher die Einführung einer Wirtingerschen Übertragung. Diesem Falle entspricht es bei Wirtinger, dass v_α noch von η^α und x^i abhängt. Nachdem Verf. die Wirtingersche Übertragung unter dieser Nebenbedingung studiert, wendet er die gefundenen Ergebnisse auf den Kawaguchischen Raum an, und zeigt, in welcher Beziehung die von Kawaguchi gegebene Übertragung mit der Wirtingerschen steht.

O. Varga (Debrecen).

Suguri, Tsuneo. On pseudo harmonic tensor fields. Mem. Fac. Sci. Kyūsyū Univ. A. **7**, 61–68 (1953).

The author gives corrections to a paper by Bochner [Ann. of Math. (2) **50**, 77–93 (1949); these Rev. **10**, 571]. In a note added in proof he remarks that, independently, similar corrections had already been made by Bochner and Yano [ibid. **56**, 504–519 (1952); these Rev. **14**, 904].

A. Nijenhuis (Princeton, N. J.).

Moreau, Jean-Jacques. Sur la structure des tenseurs isotropes et des tenseurs de révolution. C. R. Acad. Sci. Paris **238**, 441–443 (1954).

The author shows that an isotropic tensor of any order in three dimensions is a tensorial polynomial in the ϵ and δ symbols. It is claimed that previous demonstrations of this theorem, which has applications in the theory of invariance, have usually been unsatisfactory. The proof involves only algebraic calculations and proceeds by induction on the order of the tensor. An analogous theorem is stated for tensors of revolution.

J. S. Klein (Ann Arbor, Mich.).

NUMERICAL AND GRAPHICAL METHODS

Reynolds, George E. Conversion table of tangents or cotangents to sines and cosines of three decimals. Electronics Research Directorate, Air Force Cambridge Research Center, Cambridge, Mass. Tech. Rep. **53-29**, 8 pp. (18 plates) (1953).

The need for a ready reference table of low accuracy giving without interpolation the sine or cosine of any first quadrant angle whose tangent or cotangent is known arises in certain layout problems involving parallel surfaces. This need is met by the table under review. This gives a list of tangents and cotangents between two values of which the sine and cosine are constant to 3 decimal places. Thus we have a direct reading of sin (arctan u), cos (arctan u), sin (arccot u), cos (arccot u). *D. H. Lehmer.*

Hirvonen, R. A. Nutshell tables of mathematical functions for interpolation with calculating machines. Bull. Géodesique **1953**, 369–392 (1953). (German, Spanish, French and Italian summaries)

✶**Bruins, E. M. Numerieke wiskunde. [Numerical mathematics.]** Servire, den Haag, **1951**. 127 pp. 4.50 fl.

Contents are: I. Approximate computation: a) Fixed scales, b) Movable scales; II. Rational powers, polynomials: 1) Integral powers, 2) Root extraction (Newton's method, binomial theorem, and a continued fraction development of $\log_{10} 5$), 3) Approximate multiplication, 4) Polynomials (forward differences, the Sturm sequence, Newton's method, nomograms for trinomial equations), 5) Concerning the method of least squares (straight lines, with one variable or both subject to error), 6) Sets of orthogonal polynomials; III. Fundamental transcendental functions: 1) Survey, 2) Calculation of exponentials, sines and cosines, 3) Calculation of π, arctan s, 4) Calculation of logarithms (with seven root extractions obtains $2^{1/512}$; then 8 terms in the series gives its logarithm, and thence ln 2 to 21 places), 5) Development of functions in terms of exponential functions, Fourier analysis, 6) Transformation of Kummer; IV. Equations in more than one variable: 1) Survey of methods (elimination, iteration, method of Newton), 2) Linear equations, 3) Nonlinear equations; V. Numerical integration and differentiation: 1) Integration of functions of one variable (trapezoidal, Maclaurin, Simpson, Gauss), 2) Numerical differentiation and integration from tables, 3) Ordinary differential equations (Runge-Kutta), 4) Partial differential equations (Runge-Kutta for first order, Liebmann-Courant for elliptic). A supplement describes square-root extraction on a desk computer. The development is in terms of numerical examples throughout, but altogether an amazing amount of material is condensed in a very short space.

A. S. Householder (Oak Ridge, Tenn.).

✶**Grossmann, Walter. Grundzüge der Ausgleichungsrechnung nach der Methode der kleinsten Quadrate nebst Anwendungen in der Geodäsie.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953. viii+261 pp. DM 19.80.

This book gives a detailed presentation, well illustrated by numerical examples, of the classical theory of errors and adjustment of observations, intended to serve both as a general text in the subject and also as a handy book of reference for anyone needing help on a specific problem. The subject matter and especially the numerical illustrations are obviously slanted toward surveying, and particularly toward geodesy.

The first chapter, entitled fundamentals of the theory of errors, contains a discussion of the types of errors, measures of accuracy, propagation of errors, mean error, relative error, probable error, true error, observations of different accuracies, weights, Gaussian law of error, and allied topics. Chapter two, on adjustment of direct observations, explains the purpose of adjustment, and illustrates the process in the cases of observations of both equal weights and unequal weights. Chapter three on indirect observations shows how to set up the error equations for both the linear and nonlinear cases, how to solve the normal equations, how the calculation is organized. It also treats the subject of control checks, reciprocals of the weights, the mean errors of the unknowns, weight of a function of the unknowns, simultaneous calculation of the unknowns and the weights, and the construction of the final equations without intermediate steps.

In chapter four the author considers observations subject to conditions, how to set up the equations of condition, the correlation equations, normal equations, and control checks, the conditional equations and side equations in triangulation, Boltz's method of development, and indirect observations with equations of condition. Chapter five takes up solution by successive approximations, approximate representation of functions, power series and trigonometric series for representing a function, mean error of the measures of accuracy, mean error of mean error, and similar quantities. As may be expected the methods of Gauss are much in evidence. A valuable feature of the book is a separate index for the illustrative examples whereby a stranger to the text can quickly locate the numerical example which illustrates any particular situation. There are in all thirty-eight such examples. *W. E. Milne* (Corvallis, Oreg.).

Hauser, Arthur A., Jr. Geometric aspects of least squares smoothing. Proc. I. R. E. **42**, 701–704 (1954).

Tweedie, M. C. K. A modification of the Aitken-Neville linear iterative procedures for polynomial interpolation. Math. Tables and Other Aids to Computation **8**, 13–16 (1954).

Aitken [Proc. Edinburgh Math. Soc. (2) **3**, 56–76 (1932)] has given a simple algorithm the repeated use of which leads to polynomial interpolation of any desired degree and has also given a scheme of computation having special advantages. Neville's method [J. Indian Math. Soc. **20**, 87–120 (1934)] uses the same algorithm but a different scheme of computation, one that generally converges faster but is less convenient to use. The present author uses the same algorithm but with a sort of centralized scheme of computation that has some of the advantages of both Aitken's and Neville's methods. *W. E. Milne.*

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Kron, Gabriel. A method of solving very large physical systems in easy stages. Proc. I. R. E. **42**, 680–686 (1954).

Young, David. On Richardson's method for solving linear systems with positive definite matrices. J. Math. Physics **32**, 243–255 (1954).

For solving $Au+d=0$ with A positive definite, Richardson's method is to select scalars β_n so that the sequence $u_n=u_{n-1}+\beta_n(Au_{n-1}+d)$ converges as rapidly as possible.

Let $e_n = u_n - u$; $AV = V\Lambda$ with V orthogonal and Λ diagonal; $e_0 = VC$. Then

$$e_p = V\prod_1^p (I+\beta_n\Lambda)C, \quad e_p^T e_p = C^T \prod_1^p (I+\beta_n\Lambda)^2 C.$$

If exactly m iterations are to be made, only the norm of e_m requires minimization. Hence if $P_m(\nu) = \prod_1^m (1+\beta_n \nu)$ one selects $P_m(\nu)$ in terms of Chebyshev polynomials as that polynomial of degree m satisfying $P_m(0) = 1$ whose maximum absolute value over an interval which contains the proper values is minimized. Rates of convergence and the effect of rounding errors are discussed, in comparison with other methods. The author's discussion presupposes that in this method the $u_1, u_2, \cdots$ are actually formed. However $u_m = P_m(A)u_0 + [P_m(A) - I]A^{-1}d$, whence u_0 and d are multiplied by polynomials in A with rational coefficients which do not require the β's explicitly. [Cf. Birman, Uspehi Matem. Nauk (N.S.) **5**, no. 3(37), 152–155 (1950); these Rev. **12**, 32.] *A. S. Householder* (Oak Ridge, Tenn.).

Hestenes, Magnus R., and Stiefel, Eduard. Methods of conjugate gradients for solving linear systems. J. Research Nat. Bur. Standards **49** (1952), 409–436 (1953).

Let A be positive definite and $Ah = k$; P_m be n by m and such that $P_m^T A P_m = D_m^2$ is diagonal and nonsingular. Let x_0 be an arbitrary vector, and form $x_m = x_0 + P_m a_m$ with some a_m. If $r_i = k - Ax_i$ $(i = 0, 1, \cdots, m)$, it is possible to select a_m so that $P_m^T r_m = 0$. In fact, $a_m = D_m^{-2} P_m^T r_0$. Moreover, for some $m \leqq n$, $x_m = h$. Any procedure for obtaining a sequence of columns $p_1, p_2, \cdots$, for matrices P_m defines a method of conjugate directions. For example, if U_m is n by m and of rank m, a Schmidt orthogonalization of the columns yields a unique unit upper triangular matrix V_m such that $P_m = U_m V_m^{-1}$ has the required properties. If, for $m = n$, $U_n = I$, then $P_n = V_n^{-1}$ and the method is equivalent to Gaussian elimination and Choleski's square-rooting. If $U_m = R_m = (r_0, r_1, \cdots, r_{m-1})$, one has the method of conjugate gradients. By this method it turns out quite simply that for a certain scalar β_m, $p_{m+1} = r_m + \beta_m p_m$, and that the matrix $R_m^T R_m$ is diagonal. By this method $(h - x_m)^T A (h - x_m)$ decreases monotonically as does $(h - x_m)^T (h - x_m)$, but $r_m^T r_m$ may oscillate.

This paper presents the most complete exposition of the method so far published. The sequence of iterates x_m is characterized geometrically; estimates of the residuals are obtained; a modification is presented in which the residuals r_m diminish in length; the effect of rounding is discussed; an extension is made for handling nonsymmetric matrices; the method is related to a method due to Lanczos [same J. **45**, 255–282 (1950); these Rev. **13**, 163] for computing characteristic values, and it is further related to orthogonal polynomials and continued fractions. Finally, a short numerical example is exhibited. *A. S. Householder.*

Have

***Rosser, J. Barkley. Rapidly converging iterative methods for solving linear equations.** Simultaneous linear equations and the determination of eigenvalues, pp. 59–64. National Bureau of Standards Applied Mathematics Series, No. 29. U. S. Government Printing Office, Washington, D. C., 1953. **$1.50.**

Consider a family of similar, similarly oriented, concentric ellipsoids. From any point x_0, choose a direction s_1 and proceed in this direction to x_1 on the hyperplane P_1 conjugate to s_1. In general, take s_i parallel to $P_1, \cdots, P_{i-1}$, proceed in that direction to x_i on the hyperplane P_i conjugate to s_i. Then x_n is the common center. This notion leads to a variety of methods for solving a system of linear algebraic equations, all iterative and leading (except for rounding errors) to the exact solution in at most n steps. [For greater detail see the paper reviewed above.] *A. S. Householder.*

van Rooijen, J. P. On numerical integration. Verzekerings-Arch. Actuarieel Bijvoegsel **30**, 41*–53* (1953).

The paper deals with numerical integration formulas of the type $\int_a^b f(x)dx = \sum_{i=0}^n p_i f(x_i)$ and especially in the case $x_k = k \cdot h$ with h as a constant. The weights p_i are derived from substitution of $f(x)$ by Lagrange's interpolation polynomial of degree n which coincides with $f(x)$ in the pivotal points x_i. The results do not differ from what is already known on the so-called method of parabolic integration [see L. Schrutka, Leitfaden der Interpolation, Springer, Wien, 1941, pp. 34–36]. *H. Bückner* (Schenectady, N. Y.).

Zemanian, Armen H. An approximate method of evaluating integral transforms. J. Appl. Phys. **25**, 262–266 (1954).

The evaluation of integral transforms described in this paper is based on an approximation of the function to be transformed by a sectionally linear function.

A. Erdélyi (Pasadena, Calif.).

Brodskiĭ, M. L. Asymptotic estimates of the errors in numerical integration of systems of ordinary differential equations by difference methods. Doklady Akad. Nauk SSSR (N.S.) **93**, 599–602 (1953). (Russian)

The notation here is not entirely clear to this reviewer. However, the difference method for solving the equations $dy/dx = f(x, y)$, where y and f represent vectors, is written in the general form

$$y_n - \sum_1^k \alpha_j y_{n-j} = h \sum_1^k \beta_j f_{n-j},$$

and the truncation error is sought. There is a lemma, a theorem, and a corollary, the latter giving the error in terms of the solution of a system of linear differential equations. The result is applied to some particular methods.

A. S. Householder (Oak Ridge, Tenn.).

Lozinskiĭ, S. M. On equations in variations. Doklady Akad. Nauk SSSR (N.S.) **93**, 621–624 (1953). (Russian)

If the vectors $y(t)$ and $Y(t)$ represent neighboring solutions on the interval $[t_0, T]$ of a system of differential equations, and $\eta(t)$ a solution of the equations for variations with $\eta(t_0) = Y(t_0) - y(t_0)$, the problem considered is to set bounds for the vector $Y - y - \eta$. The method closely parallels that used in a previous paper by the same author [same Doklady (N.S.) **92**, 225–228 (1953); these Rev. **15**, 473]. The theorems are analogues of those developed in the previous paper. *A. S. Householder* (Oak Ridge, Tenn.).

Wagner, Carl. On the numerical solution of Volterra integral equations. J. Math. Physics **32**, 289–301 (1954).

The integral equation $y(x) = f(x) - g(x)\int_0^x K(x, z)y(z)dz$ is numerically solved by means of a step by step method for the arguments $x = kh$ with $h =$ const. and with $k = 0, \frac{1}{2}, 1, 2, \cdots$. Let F_k be the approximation to $y(kh)$. Then the method applied is to calculate first $F_{\frac{1}{2}}$ by means of the Neumann-Liouville iteration or in another manner. A polynomial $p(z)$ of second order, coinciding with F_0, $F_{\frac{1}{2}}$, F_1 in the pivotal points, is substituted for $y(z)$, thus leading to

$$F_1 = f(h) - g(h)\int_0^h K(x, z)p(z)dz = f(h) - g(h) \times (A_{1,0}F_0 + A_{1,\frac{1}{2}}F_{\frac{1}{2}} + A_{1,1}F_1),$$

a relation which allows the evaluation of F_1. This procedure is extended to relations

$$F_n=f(nh)-g(nh)\int_0^{nh}K(x,s)p(s)ds$$
$$=f(nh)-g(nh)\sum_{k=0}^{n}A_{nk}F_k-\epsilon g(nh)A_{n,\frac{1}{2}}F_{\frac{1}{2}}$$

with $\epsilon=0$ when n even and $\epsilon=1$ otherwise. $p(s)$ is a polynomial of second order in the intervals $(0, 2h)$, $(2h, 4h)$, etc. when n is even, and in the intervals $(0, h)$, $(h, 3h)$, $(3h, 5h)$, etc. when n is odd; $p(s)$ coincides with F_k in the pivotal points kh. The author refers to similar methods, already known, which use linear approximation of $Y(s)$ or parabolic approximation for the whole integrand. He gives formulas for the coefficients A_{nk}, especially for the case $K(x, s)=K(x-s)$, and presents a numerical example related to a heat conduction problem. *H. Bückner.*

★Rohrberg, Albert. Theorie und Praxis der Rechenmaschinen. B. G. Teubner Verlagsgesellschaft, Stuttgart, 1954. 72 pp. DM 3.90.

Stock, John R. An arithmetic unit for automatic digital computers. Z. Angew. Math. Physik 5, 168–172 (1954).

Gibellato, Silvio. La macchina calcolatrice analogica elettrica "G. A. Philbrick". Univ. e Politecnico Torino. Rend. Sem. Mat. 12, 53–66 (1953).

Robb, James C. A calculator for aiding matrix calculations. Trans. Faraday Soc. 50, 8–12 (1 plate) (1954).

Ham, J. M. A computer for solving integral formulations of engineering problems by methods of successive approximations. Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Mass., Tech. Rep. No. 241, ii+54 pp. (1953).

Blanc, André. Dispositif à piles et relais permettant la résolution analogique d'équations aux dérivées partielles non linéaires au moyen d'une chaîne à résistances et capacités. C. R. Acad. Sci. Paris 238, 1377–1378 (1954).

Bennett, W. R. The correlatograph. A machine for continuous display of short term correlation. Bell System Tech. J. 32, 1173–1185 (1953).

Michkovitch, V. V. Rationalisateur graphique. Srpska Akad. Nauka. Zbornik Radova 35. Mat. Inst. 3, 5–10 (1953). (Serbo-Croatian. French summary)

★Walther, A., und Kron, A.-W. Nomographie und Rechenschieber. Naturforschung und Medizin in Deutschland, 1939–1946, Band 3. Angewandte Mathematik, Teil I, pp. 119–127. Verlag Chemie, Weinheim, 1953. DM 20.00.

ASTRONOMY

Mineo, Corradino. Sul modo di risolvere una indeterminazione nel problema di Clairaut generalizzato delle configurazioni degli astri fluidi rotanti in equilibrio relativo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 14, 724–727 (1953).

The stratification in a figure of equilibrium is given by the well known condition $\int_\sigma(1+\zeta)^3d\sigma=4\pi$, $r=a(1+\zeta)$, provided the density is expressed in terms of the parameter a. However, the problem becomes completely determined only if the precise definition of this parameter is also given. In author's opinion the condition used by Liapounov [Mém. Acad. Imp. Sci. St.-Pétersbourg (8) 14, no. 17 (1903)] is acceptable, another one introduced by Wavre [Figures planétaires et géodésie, Gauthier-Villars, Paris, 1932] is too arbitrary. He expects to clarify this indetermination in another paper. *W. S. Jardetzky* (New York, N. Y.).

★Happel, H. Das Dreikörperproblem. Vorlesungen über Himmelsmechanik. K. F. Koehler Verlag, Leipzig, 1941. xi+526 pp.

In this book is to be found a rather full, almost verbose, treatment of most fundamentals of the three-body problem. The first three chapters are mostly of an introductory nature treating of such topics as the equations of motion of the n-body problem, Lagrange's equations, Hamilton's equations, canonical transformations, Jacobi's partial differential equation, with applications to the Keplerian motions, the theory of whose perturbations makes up a large part of the subsequent developments.

The next two chapters are largely concerned with the perturbations by large planets of small planets. This includes not only the restricted problem of three bodies but also the more general problems where the orbit of the large planet is elliptical and where the motion of the small planet is not confined to the orbital plane of the large one. Attention is largely confined to first-order perturbations.

Chapters VI and VII deal with lunar theory, which though basically similar to the problem of small planets, permits far more extensive developments because of the characteristic simplification resulting from the neglect of parallax (i.e. the ratio of the moon's distance from the earth to its distance from the sun). Here again the treatment is not confined to the classical two-dimensional case of Hill, although in connection with the latter should be mentioned an elegant geometrical-mechanical proof of the existence of the Hill periodic solution. This existence proof is complemented by appropriate solutions in the form of infinite series. The author's own contributions are at a maximum in these two chapters. To underline the connection between his work and the well known work of Hill and Brown, the author's preface may be quoted as follows:

"Zunächst glaube ich sagen zu können, dass die Grundgedanken der Hillschen Untersuchungen sich wiederfinden—wenn auch in veränderter Fassung—in der in Kapitel VII entwickelten Theorie, eine gewisse Ähnlichkeit zwischen ihr und der von Hill wird der Kundige erkennen. Dazu kommt noch, dass die Störungen, falls man sie gemäss den Vorschriften von Kapitel VII berechnet, sich in einfacherer Weise ergeben wie bei Hill-Brown. Insbesondere hat A. Karras auf meine Anregung in seiner Dissertation die von der Mondexzentrizität abhängigen Störungen ermittelt." Further comments along these lines are given on page 339, including a reference to work of Karras.

In Chapter VIII the author gives an account of periodic perturbations, concerned with Hamiltonian systems in which $H=H_0+\mu H_1+\mu^2H_2+\cdots$, where H_0 depends only on x_1, while the other H_i depend on x_1, x_2, y_1, y_2 and y' and are periodic with respect to the latter three variables with

period 2π while y' is a given linear function of t. All this, of course, is based on Poincaré. The author also gives the slight modification used by E. Hopf in proving the existence of the Hill periodic solution in lunar theory, even though this subject is also treated by another method in the preceding chapter.

The characteristic exponents of periodic solutions are introduced first in Chapter IX. Much in this chapter probably could have been presented more effectively from a more general point of view. For instance the theorem about the occurrence of these exponents in pairs (of opposite sign and equal magnitude) is stated only for the restricted problem of three bodies, when actually it is true for all Hamiltonian systems (or more generally yet of all Pfaffian systems). Chapter X gives an account of the particular triangle and straight line solutions of Lagrange, as well as motion in the neighborhood of a libration point and the problem of Hecuba. The book closes with a short chapter on the regularization of the three-body problem in the neighborhood of binary collision and the theorem of Sundman to the effect that triple collision is impossible unless the angular momentum is zero.

For the mathematician the book should prove useful as a convenient reference to the formal manipulations of the astronomers, which often pose extremely interesting questions of a more purely mathematical nature. In this connection it is to be regretted that there is no bibliography, but only a few scattered references given in parentheses through the text. *D. C. Lewis* (Baltimore, Md.).

Radzievskiĭ, V. V. General solution of a case of the problem of three bodies. Doklady Akad. Nauk SSSR (N.S.) 91, 1309–1311 (1953). (Russian)

Consider a homogeneous spherical cosmic cloud of constant density and two quasi-particles moving inside this cloud under the action of their mutual attraction and attraction toward the center of the cloud. The author shows that, if the density of the cloud is sufficiently small, so that one may neglect the resistance of the medium, the problem of relative motion of the two bodies can be solved by quadratures for arbitrary initial conditions.

E. Leimanis (Vancouver, B. C.).

Heinrich, Wladimír Wáclav. On a surprising possibility in the Lunar theory. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1951, no. 13, 30 pp. (1953).

This paper is identical with part I of a previously reviewed paper [Acta Math. 88, 1–75 (1952); these Rev. 14, 590]. *E. Leimanis* (Vancouver, B. C.).

Heinrich, Wladimír Wáclav. On new particular integrals of the satellite problem of three bodies. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1952, no. 4, 45 pp. (1953). (Czech summary)

This paper is identical with part II of a previously reviewed paper [Acta Math. 88, 1–75 (1952); these Rev. 14, 590]. *E. Leimanis* (Vancouver, B. C.).

Sretenskiĭ, L. N. The motion of three particles on rotating orbits. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 8, 15–19 (1953). (Russian)

Newton [Principia, Book I, Sect. IX, Dawson, London; these Rev. 14, 833] proposes to find a central force such that under its action a particle moves in a curve which revolves about the center of force in the same manner as another particle in the same curve at rest. A geometrical analysis showed that the required central force differs from the inverse square law by a force varying as the inverse cube of the distance. The author shows that by addition of such a force to the force of Newtonian attraction, three particles may be made to describe in a plane revolving about their common center of gravity the same orbits as Lagrange's equidistant particles would describe in the fixed plane under the action of Newtonian attraction alone. *E. Leimanis.*

Mihailović, Dobrivoje. Su un metodo più generale per la riduzione del problema di due corpi colle masse permutabile al problema di due corpi della meccanica celeste. Bull. Soc. Math. Phys. Serbie 5, no. 1–2, 67–76 (1953). (Serbo-Croatian. Italian summary)

The author generalizes the method of Batyrev [Akad. Nauk SSSR. Astr. Žurnal 26, 56–59 (1949); these Rev. 10, 577] for reducing the problem of two bodies with variable masses to the classical two-body problem in celestial mechanics. Using two theorems of Dubošin [ibid. 9, 53–56 (1932)] and the fact that the total mass $M=m_1+m_2=f(t)\to 0$ as $t\to\infty$, the author concludes that the orbit is neither periodic nor asymptotic in the finite region of motion; moreover, it is impossible to find initial conditions for which the trajectory approaches an ellipse as its limiting orbit. The qualitative analysis of Batyrev concerning the shape of the trajectories is confirmed. *E. Leimanis.*

***Vidal Abascal, E. Calculo de orbitas de estrellas dobles visuales. [Calculation of the orbits of visual binary stars.]** Monografias de Astronomia y Ciencias Afines, No. 1. Consejo Superior de Investigaciones Cientificas. Santiago, 1953. 218 pp. (1 plate)

This volume contains much material on the computation of double-star orbits that has heretofore been accessible only in original publications. An introductory chapter gives a historical summary of the subject, a listing of the principal catalogues of double star observations and a brief description of the character of numerous methods that have been proposed for deriving the relative orbit from observations. Chapter II deals with the relation between the relative orbit and the apparent orbit and with properties of elliptic motion. Chapters III and IV present in detail various methods with complete examples. Particular attention is given to the Thiele-Innes method in which the relative orbit is obtained from observations without first deriving the apparent orbit. Chapter V deals with ephemeris calculations, chapter VI with methods for orbit correction. An appendix by Ramon M. Aller gives information on methods of observation and reduction. Five auxiliary tables, occupying 50 pages, and an extensive bibliography conclude the volume.

D. Brouwer (New Haven, Conn.).

Gomes, Alércio Moreira. On the canonic constants of a planetary orbit. Revista Científica 3, no. 1–2, 3–6 (1952).

An expression for the radius vector obtained in the course of the integration of the Hamilton-Jacobi differential equation for the two-body problem is identified with the expression for the radius vector as a function of the true anomaly. Thus the geometrical meaning of one of the canonical elements is obtained. [Errors in equations (1), (8), and (9) are typographical and do not affect the derivation.]

D. Brouwer (New Haven, Conn.).

Kushwaha, R. S. Anharmonic oscillations of a particular model. Bull. Calcutta Math. Soc. **45**, 75–81 (1953).

The author considers the anharmonic radial oscillations of a stellar model with two thirds of the total mass homogeneously distributed and the remaining third concentrated in a point mass at the center, and compares the results with his earlier results [Proc. Nat. Inst. Sci. India **17**, 323–329 (1951); these Rev. **13**, 498] for the Roche model. The computations indicate that there is little difference in the skewness and period increase in the two models. In addition the author studies the effects of the variation of the semi-amplitude and of the effective ratio of specific heats on the skewness and period of his present model.

R. G. Langebartel (Urbana, Ill.).

RELATIVITY

Lichnerowicz, André. Compatibilité des équations de la théorie unitaire d'Einstein-Schrödinger. C. R. Acad. Sci. Paris **237**, 1383–1386 (1953).

The new unified theory of Einstein is based on the conditions

$$(1)\qquad g_{\underset{+}{\lambda}\underset{-}{\mu};\omega}=0,\quad \Gamma_\lambda=0,\quad R_{(\lambda\mu)}=0,\quad \partial_{[\omega}R_{\mu\lambda]}=0.$$

This system of differential equations has been solved in some special cases [for an (incomplete) bibliography of these special solutions cf. the paper reviewed below]. In this paper the author announces a theorem concerning the solution of (1) based on initial conditions (along a hypersurface $x^0=0$) alone, without any other restrictions. Such a solution is unique. The detailed proof will be published later. (There is a misprint in the definition of the tensor $H_\rho{}^\lambda$. The left-hand term of the corresponding equation should read $2H_\rho{}^\lambda$.)

V. Hlavatý (Bloomington, Ind.).

Hlavatý, Václav. The elementary basic principles of the unified theory of relativity. C₁. Introduction. J. Rational Mech. Anal. **3**, 103–146 (1954).

[For parts A and B see same J. **1**, 539–562 (1952); **2**, 1–52 (1953); these Rev. **14**, 416, 505. A closely related paper is that of Hlavatý and Sáenz, ibid. **2**, 523–536 (1953); these Rev. **14**, 1132.] In the current unified theory of Einstein, the asymmetric fundamental tensor $g_{\lambda\mu}$ and the affine connection $\Gamma^\nu{}_{\lambda\mu}$ are related by the equations

$$(1)\qquad \partial_\omega g_{\lambda\mu}=\Gamma^\alpha{}_{\lambda\omega}g_{\alpha\mu}+\Gamma^\alpha{}_{\omega\mu}g_{\lambda\alpha},\quad \Gamma^\alpha{}_{\lambda\alpha}=\Gamma^\alpha{}_{\alpha\lambda},$$

together with a set of conditions that may be condensed into

$$(2)\qquad R_{\mu\lambda}=\partial_{[\mu}X_{\lambda]},$$

$R_{\mu\lambda}$ being the contracted curvature tensor of $\Gamma^\nu{}_{\lambda\mu}$, and X_λ an arbitrary vector that may or may not be the gradient of a scalar. The present long paper, which is divided into two chapters, is wholly analytical, physical applications being reserved for later papers. Chapter I begins with the establishment of formulae necessary for the subsequent study of the bivector $m_{\lambda\nu}$ defined by

$$m_{\lambda\nu}=-\tfrac{1}{2}\mathfrak{e}_{\lambda\nu\alpha\beta}\,{}^*g^{[\alpha\beta]}\sqrt{|\mathfrak{g}|},$$

where $\mathfrak{e}_{\lambda\nu\alpha\beta}$ is the fundamental alternating weighted tensor of components $(\pm 1, 0)$, $\mathfrak{g}=\det\|g_{\lambda\mu}\|$ and ${}^*g^{\alpha\beta}$ is the contravariant tensor defined by ${}^*g^{\nu\mu}g_{\lambda\mu}=\delta_\lambda^\nu$. This bivector is to define the electromagnetic field in future papers. The later part of the chapter is concerned with the curvature tensor and with a tensor $g_{\lambda\mu}$ having special properties, the formulae obtained being required for subsequent purposes. In Chapter II the author deals with a conformal change of $g_{\lambda\mu}$, with path-preserving transformations

$$\Gamma^\nu{}_{\lambda\mu}={}^\circ\Gamma^\nu{}_{\lambda\mu}+\delta^\nu{}_\lambda p_\mu+\delta^\nu{}_\mu p_\lambda$$

of the affine connection, and with existence problems. Using a particular solution, he builds up the most general solution of the systems of equations (1), (2), the method being one that enables him to pick out solutions that might be suitable for physical applications. The chapter concludes with a particular case which will be treated from the physical point of view in the next of the three further papers promised by the author.

H. S. Ruse (Leeds).

Hlavatý, Václav. The elementary basic principles of the unified theory of relativity. C₂. Applications. I. J. Rational Mech. Anal. **3**, 147–179 (1954).

[For parts A, B, C₁ of this investigation, see same J. **1**, 539–562 (1952); **2**, 1–52 (1953); these Rev. **14**, 416, 505; and the paper reviewed above. See also Hlavatý and Sáenz, ibid. **2**, 523–536 (1953); these Rev. **14**, 1132.] The affine connection of the unified theory may be expressed in the form

$$\Gamma^\nu{}_{\lambda\mu}=\left\{{\nu \atop \lambda\mu}\right\}+S_{\lambda\mu}{}^\nu+U^\nu{}_{\lambda\mu},$$

where the Christoffel symbols are formed from the symmetric part $h_{\lambda\mu}$ of the fundamental tensor $g_{\lambda\mu}$, and $S_{\lambda\mu}{}^\nu$ is the torsion-tensor for which, by assumption, $S_{\lambda\alpha}{}^\alpha=0$. In this paper the author takes the special case in which $U^\nu{}_{\lambda\mu}=0$, a condition which imposes a structural restriction on $g_{\lambda\mu}$ and which is automatically satisfied for weak electromagnetic fields. He reserves the case $U^\nu{}_{\lambda\mu}\neq 0$ for a later paper. He obtains the equations

$$H_{\mu\lambda}-\tfrac{1}{2}Hh_{\lambda\mu}=T_{\lambda\mu},$$

where $H_{\mu\lambda}$ is the contracted curvature-tensor of $h_{\lambda\mu}$, and $T_{\lambda\mu}$ is a well-defined function of all arguments that enter into the total momentum-energy tensor of gravitational theory in the presence of an electromagnetic field. The tensor $T^{\lambda\nu}$ satisfies the equation $D_\lambda T^{\lambda\nu}=0$, where D_λ denotes covariant differentiation with respect to $\Gamma^\nu{}_{\lambda\mu}$, and from this is obtained an equation

$$w^\lambda D_\lambda w^\nu=P^\nu,$$

the form of which, as well as the method of obtaining it, suggests that it is to be interpreted as giving the equations of motion of a photon or of a particle according as the vector w^λ (in the language of general relativity, not used by the author and not here really appropriate) is null or non-null. Possible observational tests are discussed. Towards the end of the paper the author replaces the Einstein equations $\partial_{[\omega}R_{\mu\lambda]}=0$ by another set of four differential equations expressed in terms of the torsion-tensor, and obtains the equations of motion

$$w^\lambda D_\lambda w^\nu=\sigma w^\nu.$$

From these he concludes that a particle moving in the $g_{\lambda\mu}$-field under no external forces describes autoparallel lines of $g_{\lambda\mu}$, which are identical with the geodesics of $h_{\lambda\mu}$. The unified theory thus obtained yields the same results for the three "crucial tests" as general relativity, but these tests are not crucial for the unified theory.

This review gives only a bare indication of the kind of work the paper contains, and is based to a large extent upon the author's own synopsis. Out of his investigation, which, including the present paper, has already occupied 155 pages,

may emerge a satisfactory unified theory. That is evidently the author's hope, and he guides the reader, as well as anyone could, through the mass of formulae, assumptions, calculations and references-back with which a theory based upon asymmetric $g_{\lambda\mu}$ and $\Gamma^{\nu}{}_{\lambda\mu}$ inevitably abounds. To the reviewer it seems, however, that the theory must be shown capable of drastic simplification if it is to gain any acceptance. The question whether any significant advance in physical theory is likely to be made on these lines at all is perhaps a matter of personal prejudice. *H. S. Ruse.*

Udeschini, Paolo. Successiva linearizzazione delle ultime equazioni del campo unitario einsteiniano. Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15**, 165–170 (1953).

1. The Einstein new unified field theory requires first the knowledge of the underlying connection $\Gamma^{\nu}{}_{\lambda\mu}$ in tensorial form, expressed in terms of the basic tensor $g_{\lambda\mu}$. This connection enables us to express its contracted curvature tensor $R_{\mu\lambda}$ by means of $g_{\lambda\mu}$. Moreover, a skew-symmetric tensor $m_{\lambda\nu}$ may be defined by means of $g_{\lambda\mu}$ which plays the role of the electromagnetic field. [Cf. the paper reviewed second above.]

2. Substituting the above mentioned results in the Einstein conditions

$$(1) \qquad R_{(\mu\lambda)}=0, \quad \partial_{[\omega}R_{\mu\lambda]}=0, \quad S_{\lambda}=0$$

(where $S_{\lambda}=S_{\lambda\alpha}{}^{\alpha}$, $S_{\lambda\mu}{}^{\nu}=\Gamma^{\nu}{}_{[\lambda\mu]}$), one converts this purely geometrical set of 18 conditions into a purely physical set of 18 conditions for the unknowns $g_{(\lambda\mu)}$, $m_{\lambda\mu}$ and v^{ν} (=the world current density). [Cf. the paper reviewed above.] The last of (1) is equivalent to the first set of Maxwell's equations $\partial_{[\omega}m_{\mu\lambda]}=0$ so that $m_{\lambda\mu}=\partial_{[\lambda}m_{\mu]}$ represents only four unknowns. Hence, we have 18 conditions (1) for 18 unknowns $g_{(\lambda\mu)}=h_{\lambda\mu}$, m_{λ}, v^{ν}.

3. The problem is very much simplified if we confine ourselves (with the author) to the tensor

$$g_{\lambda\mu}=\overset{0}{h}_{\lambda\mu}+\epsilon\overset{1}{g}_{\lambda\mu}+\epsilon^2\overset{2}{g}_{\lambda\mu}$$

where $\overset{0}{h}_{\lambda\mu}=\overset{0}{h}_{(\lambda\mu)}$ and ϵ are constants and $\epsilon^3\to 0$. In this case the first two equations (1) are equivalent to

$$\Box\epsilon^2\overset{2}{g}_{(\lambda\mu)}=B_{(\lambda\mu)}\epsilon^2, \quad \Box\epsilon^2\partial_{[\omega}\overset{2}{g}_{\lambda\mu]}=\partial_{[\omega}B_{\lambda\mu]}\epsilon^2,$$

where $B_{\lambda\mu}$ is well defined in terms of $\epsilon\overset{1}{g}_{\lambda\mu}$ and $\Box$ is the usual symbol of d'Alembert with respect to $\overset{0}{h}_{\lambda\mu}$. *V. Hlavatý.*

Clauser, Emilio. Una particolare soluzione delle equazioni einsteiniane della relatività unitaria. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15**, 171–177 (1953).

The 82 equations of the new unified field theory of Einstein have been solved for some special cases [cf. Papapetrou, Proc. Roy. Irish Acad. Sect. A. **52**, 69–86 (1948); these Rev. **10**, 580; Bandyopadhyay, Indian J. Phys. **25**, 257–261 (1951); these Rev. **13**, 994; and Takeno, Ikeda, and Abe, Progress Theoret. Physics **6**, 837–848 (1951); these Rev. **13**, 787; see also the paper reviewed third above]. The author solves the aforementioned system in the special case

$$g_{\lambda\mu}=\begin{Vmatrix} h_{11} & 0 & 0 & k_{14} \\ -0 & h_{22} & 0 & 0 \\ 0 & 0 & h_{33} & 0 \\ +k_{41} & 0 & 0 & h_{44} \end{Vmatrix}$$

where all the h's as well as $k_{14}=-k_{41}$ are functions of one variable only. *V. Hlavatý* (Bloomington, Ind.).

Winogradzki, Judith. Sur les géodésiques de l'Univers d'Einstein-Schrödinger. C. R. Acad. Sci. Paris **238**, 996–998 (1954).

The author studies the following problem: What are the conditions on the space of Einstein's latest unified field theory (asymmetric metric tensor and asymmetric affine connection), in order that the unit tangent vectors of geodesics be propagated parallelly along them? *A. E. Schild* (Pittsburgh, Pa.).

Tharrats Vidal, Jesús M.ª. Significance of the new field of Einstein. An. Real Soc. Españ. Fis. Quim. Ser. A. Fis. **49**, 303–310 (1953). (Spanish. English summary)

In this work it is shown that the metric and non-symmetrical connection of the new generalized gravitational field recently introduced by Einstein (1946–50) has as local metric the group of projectivities that leave invariant a correlation. Such spaces have not only the property of being anisotropic, but are also non-commutative, that is to say, that they fulfill the inequality $\widehat{ab}\neq\widehat{ba}$. (Author's summary.) *H. P. Robertson* (Pasadena, Calif.).

Finzi, Bruno. La recente teoria relativistica unitaria di Einstein. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11**, 75–87 (1952).

Lecture given in April 1951.

Finzi, Bruno. Lo spazio-tempo come modello dei fenomeni fisici. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11**, 62–74 (1952).

Lecture given in April 1951.

De Donder, Th. Sur les quatre liaisons introduites dans la gravifique einsteinienne. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 1024–1026 (1953).

Hély, Jean. Sur une représentation du champ unitaire. C. R. Acad. Sci. Paris **238**, 1375–1377 (1954).

The author starts with a metric non-symmetric connection

$$\Gamma^{\nu}{}_{\lambda\mu}=\left\{ {\nu \atop \lambda\mu} \right\}+G_{\lambda\mu}{}^{\nu}$$

so that $G_{\lambda\mu\nu}$ must be skew-symmetric in all three indices. Imposing some conditions on $\Gamma^{\nu}{}_{\lambda\mu}$ which require the existence of covariant constant vectors, he obtains field equations which reduce either to a system closely connected with de Broglie-Proca equations or with Born-Infeld equations. [Remark of the reviewer: Since the existence of covariant constant vectors is an extremely strong restriction of generality, it remains to be seen whether or not this restriction and the above mentioned system are physically compatible and meaningful.] *V. Hlavatý.*

Ueno, Yoshio. On the wave theory of light in general relativity. I. Path of light. Progress Theoret. Physics **10**, 442–450 (1953).

After a discussion of the known facts that in general relativity the wave equation $g^{ij}\psi_{,ij}=0$ (comma denotes covariant differentiation) has for characteristics the 3-spaces $\phi=\text{const.}$ where $g^{ij}\phi_{,i}\phi_{,j}=0$ and for bicharacteristics the null geodesics contained in these 3-spaces, an appendix (of which H. Takeno is joint author) gives proofs of some properties of congruences of null lines defined by a vector field v^i ($v^iv_i=0$). Among the results is the following. Let the congruence consist of null geodesics, the field v^i being normalized so that the equations of the geodesics are

$v^i{}_{,j}v^j=0$. Then a necessary and sufficient condition that this congruence be normal is $P_{ij}P^{ij}=0$, where $P_{ij}=v_{i,j}-v_{j,i}$.
J. L. Synge (Dublin).

Heckmann, O., Jordan, P., und Fricke, W. Zur erweiterten Gravitationstheorie. I. Z. Astrophys. **28**, 113–149 (1951).

One of the fundamental papers in the development of the Hamburg projective relativity, in which the constant of gravitation is taken as a field variable. This theory was presented in P. Jordan's "Schwerkraft und Weltall" [Vieweg, Braunschweig, 1952; these Rev. **14**, 1022].

The present paper is concerned with the Schwarzschild problem of the spherically symmetric field, on the basis of alternative field equations A and B. The solution of the former set is found unsatisfactory; that obtained from the field equations B, which are those adopted in the later form of the theory, is that given in Chap. IV of "Schwerkraft und Weltall". *H. P. Robertson* (Pasadena, Calif.).

I.
Frankl', F. ~~S~~. Some remarks on principles in the general theory of relativity. Uspehi Matem. Nauk (N.S.) **8**, no. 3(55), 160–164 (1953). (Russian)

The author discusses some of the concepts of the general theory of relativity and attempts to justify them from the Marxist standpoint. On the other hand he criticizes some of the ideas of Einstein and other "Machist" physicists from the same point of view. *N. Rosen* (Haifa).

Scheidegger, A. E. Gravitational motion. Rev. Modern Physics **25**, 451–468 (1953).

This review paper is concerned with two problems in the general theory of relativity: the problem of the motion of two or more bodies in gravitational interaction, and the problem of gravitational radiation.

Unlike classical field theories (such as Maxwell-Lorentz electrodynamics), the general theory of relativity (1) is non-linear, and (2) does not admit any space-time frames which are assigned a priori as having preferred physical properties (such as inertial frames). It is these facts which make the problems of motion and of radiation in general relativity both more difficult and more interesting than in classical physics.

In Einstein's original formulation in 1916 of the general theory of relativity there appear, side by side with the field equations, equations of motion for a mass particle, the geodesic equations

$$(*)\qquad \frac{d^2x^\mu}{ds^2}+\left\{\begin{matrix}\mu\\ \alpha\beta\end{matrix}\right\}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0.$$

On the world line of a point particle, the gravitational field tensor or metric $g_{\mu\nu}$ is singular; thus neither ds nor the Christoffel symbols $\{{}^{\mu}_{\alpha\beta}\}$ exist there. This raises the question: How can it be verified whether or not a world line satisfies the equation (*), if at every point on the curve none of the terms of the equation exist? This difficulty is not new and is present in classical field theories also. There it is resolved (in part) by inserting in the equations of motion of a particle a regular background field from which the singular self field of the particle has been subtracted. In general relativity, it is the non-linearity of the theory which spoils this method of approach: the superposition principle of fields is not valid, and it is not possible to give a unique prescription which divides the total field into background and self fields.

These important difficulties can be glossed over in the special case of a single light particle moving in the gravitational field of much heavier bodies. In this one-body problem the geodesic equations (*) can be used successfully, and they lead to Newtonian orbits with the well known relativistic correction for the advance of perihelion. But the difficulties are inescapable in the many-body problem of two or more particles of comparable masses. It is therefore not surprising that until 1937 all attempts to formulate this problem remained unsuccessful, or at best unsatisfactory.

In 1937, Einstein, Infeld and Hoffmann [Ann. of Math. (2) **39**, 65–100 (1938)], using only the field equations in empty space, investigated solutions which, along some world lines, have simple pole singularities representing simple mass-particles. They showed that such solutions can exist only if the singular world lines satisfy certain differential equations, the equations of motion. Thus the equations of motion of the many-body problem were obtained by deriving them from the field equations. It is interesting to note that this feature would be undesirable in a linear field theory: By linearly superimposing solutions with different particle sources, a new solution of the field equations is obtained with unchanged particle motions. Thus, if the motion of sources were determined by linear field equations, particles could not interact at all. The same non-linearity which makes the interpretation of the geodesic equations (*) so difficult, allows us to dispense with them altogether.

The first part of Scheidegger's paper summarizes the work of Einstein-Infeld-Hoffmann and later developments. The bibliography is quite complete. One omission is to the historically interesting paper of A. Eddington and G. L. Clark [Proc. Roy. Soc. London. Ser. A. **166**, 465–475 (1938)]. These authors, independently of Einstein-Infeld-Hoffmann but one year later, obtained the correct relativistic approximation to the two-body problem by the older methods which use the geodesic equations and the energy-momentum tensor. [For a review of this approach see also T. Levi-Cività, Mémor. Sci. Math. no. **116** (1950); these Rev. **13**, 499.]

The second part of Scheidegger's paper deals with gravitational radiation. In the reviewer's opinion this important problem is still "open" and well worth further investigation. Again the main point is the formulation of a mathematically unambiguous and physically meaningful definition of the term "gravitational radiation". Because of the non-linearity of general relativity it is difficult to give a satisfactory prescription which separates the gravitational field into a self field of the sources and a radiation field. Even if such a prescription is given, the question arises whether the features which distinguish the radiation field can be transformed away by a change of coordinates. If the answer is yes, then the radiation is spurious and characterizes a particular space-time frame rather than a physical state. This is illustrated by the following development: In the ninth order of the Einstein-Infeld-Hoffmann approximation procedure, N. Hu [Proc. Roy. Irish Acad. Sect. A. **51**, 87–111 (1947); these Rev. **8**, 496] found curious radiation effects which seemed to show that accelerated particles absorb rather than emit gravitational radiation. In 1949, Infeld [L. Infeld and A. E. Scheidegger, Canadian J. Math. **3**, 195–207 (1951); these Rev. **13**, 169] showed that Hu's terms can be wiped out by a coordinate transformation and that the negative radiation damping is therefore spurious. The author argues that in a purely gravitational system, all radiation effects are spurious so that there is no gravita-

tional radiation in any physical sense. [For a different opinion see J. N. Goldberg, Physical Rev. (2) 89, 263–272 (1953), Sec. IV; these Rev. 14, 805.] *A. E. Schild.*

Maravall Casesnoves, Darío. Relativistic theory of the attraction of a sphere, pulsating or with spin. Application to the cepheids. Revista Mat. Hisp.-Amer. (4) 13, 175–187 (1953). (Spanish)

An investigation of general relativistic fields and equations of motion for rotating and for pulsating spheres. The fields are obtained with the aid of the approximate solution of the n-particle problem, as given by Chazy (in which, be it noted, the coefficient of the second term in the 2nd order approximation should be $+2$ instead of -4). The author re-derives Thirring's interior and exterior fields for rotating spherical shells, and sets up the 1st order equations for the motion of a particle therein. The same methods are applied to the problem of a radially pulsating spherical shell. The author considers the mixed space-time coefficients of the metric as analogues of the components of the vector potential of an electromagnetic field, as they are responsible for a (Coriolis-like) force which is orthogonal to the velocity of the moving particle. *H. P. Robertson* (Pasadena, Calif.).

Lalan, Victor. Les transformations de Lorentz forment-elles un groupe? Ann. Physique (12) 8, 653–661 (1953).

The author shows that the set of special Lorentz transformations, for which the components v_i' of the velocity of S relative to S' are the negatives $-v_i$ of those of S' relative to S, do not form a group (which is responsible, in the case of infinitesimal special Lorentz transformations, for the Thomas precession). *H. P. Robertson.*

Vescan, Teofil T. Contributions à la critique de la cosmologie relativiste. Acad. Repub. Pop. Române. Bul. Şti. Sect. Şti. Mat. Fiz. 3 (1951), 561–568 (1952). (Romanian. Russian and French summaries).

An attempt to find a model for relativistic cosmology in which matter is converted into radiation in accordance with the hypotheses

$$M=M_0+M_1e^{-\nu t},\quad P=P_0(e^{\nu t}-1),$$

where M is the mass and P the pressure of radiation at time t. The author concludes that such a model must oscillate between two finite limits, and that it can represent only a cluster of galaxies. Quoting, "It is thus proven that modern physics can not lead to finite solutions for the physical universe, solutions which would not conform to dialectical materialism, being in contradiction with the scientific conception of the world". *H. P. Robertson.*

Brahmachary, R. L. On the derivation of Friedmann's solution for a new cosmological model. I, II. Naturwissenschaften 41, 82–83, 136 (1954).

The gravitational field equations are derived for a model of an expanding universe containing an electrically charged fluid. *A. E. Schild* (Pittsburgh, Pa.).

García, Godofredo. New investigations and results "On the expanding universe and the origin of nebulae." Actas Acad. Ci. Lima 16, 3–44 (1953). (Spanish)

An investigation of models of the expanding universe in which a term identified as pressure is set equal to the density. The author claims to obtain therewith a satisfactory account of the velocity-distance relationship, the temperature of nebulae, and the time-scale ($\sim 3.33\times 10^9$ years) of the universe. *H. P. Robertson* (Pasadena, Calif.).

Kirkwood, Robert L. The physical basis of gravitation. Physical Rev. (2) 92, 1557–1562 (1953).

The author proposes a theory of gravitation, based on Euclidean geometry, in which the gravitational field asserts itself by determining the state of motion of inertial frames at each point of the field. The velocity field of such an inertial frame is suggestive of an "ether flow", but no conflict is seen between this representation and the special theory of relativity, which is in fact called upon to supply formulae for Doppler effect and for the variation of mass with velocity relative to the preferred state of motion. The equations of motion are then taken in Newtonian form, asserting that the rate of change of momentum relative to the inertial frame is equal to the impressed force other than gravitation—and is accordingly zero, as in the general theory of relativity, if gravity is the only influence to which the particle is subjected.

For the case of a single spherically symmetric center of gravitation, the velocity field of an inertial frame is given by the radial parabolic velocity at points of the field, in accordance with the hypothesis that a small test mass, falling freely from infinity, suffers no absolute acceleration. On the basis of this determination of the inertial field associated with the Sun, and with the aid of general physical assumptions concerning light and matter such as those mentioned above, the author derives the mass red-shift in light from the Sun, the deflection of light at the limb of the Sun, and the advance of the perihelion of a planet, all in agreement with Einstein's predictions on the basis of the general theory of relativity. Hamilton's principle for the orbit problem is shown in fact to be mathematically identical with the requirement that the orbits be geodesics of a line element which is equivalent to that of Schwarzschild. *H. P. Robertson* (Pasadena, Calif.).

MECHANICS

***Hamel, G. Theoretische Mechanik.** Naturforschung und Medizin in Deutschland, 1939–1946, Band 4. Angewandte Mathematik, Teil II, pp. 1–5. Verlag Chemie, Weinheim, 1953. DM 10.00.

***McShane, Edward J., Kelley, John L., and Reno, Franklin V. Exterior ballistics.** The University of Denver Press, Denver, Colo., 1953. xii+834 pp. (1 plate). $12.00.

This long awaited volume is destined to become a classic and to take a worthy place alongside earlier notable works on ballistics. In it we have a text-book on exterior ballistics in which the subject is developed logically and rigorously from first principles, where assumptions when justifiable are justified and when not justifiable are fairly discussed, instead of the vague appeals to 'well-known' principles of mechanics, the incomplete analogies, the surreptitious dropping of awkward 'negligible' terms and the failure to separate the mathematical from the empirical which characterise many earlier books, and which are the despair of the critical reader. Because of this the book is not made harder to read but easier. Even the reader with relatively meagre mathematical equipment should be capable of understanding nearly everything if he is willing to apply himself and makes good use of the excellent first chapter, which is a

concise compendium of all the techniques which are used in the sequel.

In Chapter I, §§ 1–6, vectors are introduced, their algebra is developed and continuous and differentiable vector functions are considered. These notions are applied in §§ 7–13 to rigid motions, momentum and force, energy, angular momentum, etc. §§ 14–15 contain a brief but unusually complete account of dimensional analysis, culminating in the Buckingham Π-theorem. In § 16 the Stieltjes integral is introduced and applied in the later §§ 17–22, which deal with probability measures, expected values, cumulative distribution functions, normal and independent distributions, etc.

In Chapter II the force system on the projectile is considered. The effects of the rotation of the earth are computed in § 1. In §§ 2–5 the aerodynamic force system is considered, first for a projectile whose surface is a surface of revolution and which has zero angular velocity about its axis, and is shown to be completely describable in terms of two forces and one couple. Then an arbitrary projectile is considered. If the surface has axial symmetry of order exceeding 2, it is shown that the aerodynamic force system can be expressed in terms of five forces and five couples; of these, four forces and four couples are associated with the transverse movement of the projectile. This force and couple system is complete and, as a whole, independent of the position of the centre of gravity. It was first set up by K. L. Nielsen and J. L. Synge [Quart. Appl. Mech. **4**, 201–226 (1946) (first issued as a restricted report in 1943); these Rev. **8**, 100], who noticed that the force system employed by R. H. Fowler, E. G. Gallop, C. N. H. Lock and H. W. Richmond [Philos. Trans. Roy. Soc. London. Ser. A. **221**, 295–387 (1920)] was incomplete. The individual forces and couples vary with different centre of mass position, this being the point about which moments are supposed taken, and relations between their associated coefficients for different positions of mass-centre are given. In § 7 the general equations of motion of a symmetric projectile are formulated. In § 8 the 'normal' equations are set up; these are the equations of motion of a particle subject only to gravity and drag, and the resulting motion is a first approximation to the actual motion.

Chapter III is devoted to methods of measurement of the aerodynamic coefficients, such as the counter-chronograph and wind tunnel tests of stationary and vibrating projectiles. In Chapter IV the normal equations are considered. The variation of density with height is treated and the use of ballistic coefficients and of the various drag-functions is explained. Applications are made to bomb and aircraft artillery trajectories. Ballistic tables, firing tables and errors produced by erroneous choice of the drag function are also discussed.

Chapter V deals with approximate methods. In § 1 the normal equations are transformed into several different forms by different choices of the independent and dependent variables. In § 2, H. P. Hitchcock and R. H. Kent's modification of the Siacci methods for flat trajectories are discussed and the Siacci functions are introduced. In § 3 the advantages of K. Popoff's introduction of oblique coordinates are mentioned. In § 4 approximate formulae for the drop are given with applications to a test sample projectile. In the remaining sections other methods and approximations are given, such as Euler's method for subsonic velocities, and the accuracy of the various methods is compared.

Chapter VI is concerned with the numerical integration of differential equations. nth order finite differences are defined and a general theorem proved which contains as particular cases the Newton-Gregory forward and backward interpolation formulae, the Newton-Gauss forward and backward formulae and the formulae of Stirling, Bessel, Everett and Steffenson. Various approximate quadrature formulae are given and an existence theorem for a system of differential equations is proved. Special methods of starting the approximate solution, when the various differences are unknown, and ways of saving time when carrying out successive approximations are discussed. These methods are applied to the normal equations and numerical examples are given.

In Chapter VII differential corrections to trajectories computed by Siacci's method are considered. These are corrections to the normal trajectory due to different conditions such as the presence of wind, change of ballistic coefficient, of air density and of velocity of sound, and different initial position and velocity. In the simplest cases, such as, for example, change of ballistic coefficient, these corrections depend upon a single number, but in others, such as change of air density, they depend not upon the individual values of this density, but upon the whole set of values of the density function; i.e. the corrections are functionals not functions. In Chapters VIII and IX these more complicated corrections are considered, and it is proved that such differential effects actually exist. Methods of computing them are also developed. It is not possible here to summarise adequately how this is done.

Chapter X is devoted to bombing from aircraft and, in particular, to the study of how to make the ballistic data, assumed computed, most readily usable by the bombardier. This involves the description of a hypothetical bombsight and the procedures for correcting for various differential effects.

In Chapter XI the angular motion of the projectile is considered. The discussion is based on the equations of Chapter II and appeared earlier in a restricted report issued by the first two authors in 1944. The equations are reformulated in dimensionless form so that the independent variable is arc length in calibres and the dependent variable the tangent of the yaw. It is assumed that the normal equations provide a first approximation to the motion, and the equations are then solved approximately. The main equation is of the form $z'+z^2-r^2=0$, where r is a slowly varying function; except when r is nearly zero,

$$z=r-r'/2r, \quad z=-r-r'/2r$$

are satisfactory approximate solutions from which the yawing motion may be derived. The stability of the motion is investigated and conditions for stability obtained in the form of two inequalities. The conditions take different forms according as the projectile is spin- or fin-stabilised. The "yaw of repose" and the drift are calculated. The treatment given of these topics represents an improvement of earlier work by Nielsen and Synge. The initial motion of the projectile is also considered from the point of view of its use in spark range measurements and yaw card trials.

In Chapter XII the deviations of trajectories from the normal due to launching effects both systematic and random, such as initial yawing motion and muzzle blast, are considered. Chapter XIII gives a short account of how spark range data are reduced in order to give estimates of the aerodynamic coefficients. In Chapter XIV the equations of motion of a rocket are obtained and solved under the

assumption of constant acceleration, gravity and a restoring lift moment being the only forces considered.

The volume concludes with an admirable historical appendix describing the growth of the science of ballistics from the earliest times. Tables of various functions associated with the Gâvre drag function are also given.

Errors or misprints noted: On pp. 195, 196 in formulae for $\dot{\mathbf{p}}\cdot\mathbf{x}_1$, $\dot{p}_1$ should be followed by a minus. At the bottom of p. 202 the value 0.04 would appear to be in error. In (16) on p. 477 insert Δ after ∂ in two places. On p. 665 ξ_e as defined is e/b not $e/2b$. On p. 742 read "medieval" for "medievil". A bibliography and index of symbols would form a useful addition to the book. *R. A. Rankin.*

***Rund, Hanno. Application des méthodes de la géométrie métrique généralisée à la dynamique théorique.** Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 41–51. Centre National de la Recherche Scientifique, Paris, 1953.

The paper gives a discussion of application of Finsler Geometry to theoretical dynamics. For a dynamical system with constraints which are not necessarily holonomic nor linear, the author suggested the principle of least curvature for the determination of the trajectories. It is shown that these are the extremals of a Lagrange problem if and only if the constraints are feebly holonomic. A geometrical interpretation of the Hamilton-Jacobi equation is given. Using a definition of stability due to Synge, it is asserted that negativeness of scalar curvature implies stability.

S. Chern (Chicago, Ill.).

Efimov, M. I. On Čaplygin's equations of nonholonomic mechanical systems. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 748–750 (1953). (Russian)

Let a "sled" be a tripod whose one leg is equipped with an infinitely thin wheel, and let the tripod slide on a plane under no forces. The author offers a solution of this problem, to replace the wrong one of Čaplygin who made the not unusual error of using the distance traveled by the center of the wheel as a coordinate, and applying his form of Lagrange equations for nonholonomic constraints. It is not claimed that Čaplygin was the first to devise this error.

A. W. Wundheiler (Chicago, Ill.).

O'Brien, Stephen, and Synge, J. L. The instability of the tippe-top explained by sliding friction. Proc. Roy. Irish Acad. Sect. A 56, 23–35 (1954).

Despite the wording of the title, this paper deals primarily with the stability of a tippe-top which is idealized in that viscous friction is substituted for sliding friction. This substitution makes it possible to discuss questions of stability in terms of the locations of the roots of an algebraic characteristic equation. An argument is given which tends to show that the substitution does not affect the conclusions essentially.

Suppose that the tippe-top is in its equilibrium position, and is spinning about the vertical axis with the angular velocity ω_3. Let C and A, respectively, denote the moments of inertia about the vertical axis and a transverse axis. Let R denote the radius of curvature of the surface of the top at the point of contact with the supporting plane, and let h denote the distance of the center of curvature above the center of gravity. It is found that if $A \leqq C(1-h/R)$, there is stability. If $A > C(1-h/R)$, there is a critical angular velocity ω_0, given by the formula

$$\omega_0^2 = \frac{mgh(1-h/R)^2}{A-C(1-h/R)},$$

and there is stability or instability according as $\omega_3 < \omega_0$ or $\omega_3 > \omega_0$.

A theory of the tippe-top previously given by Synge [Philos. Mag. (7) 43, 724–728 (1952); these Rev. 14, 100] is now stated to be inadequate. *L. A. MacColl.*

Hydrodynamics, Aerodynamics, Acoustics

***Betz, Albert. Inkompressible Strömungen.** Naturforschung und Medizin in Deutschland, 1939–1946, Band 11. Hydro- und Aerodynamik, pp. 1–19. Verlag Chemie, Weinheim, 1953. DM 14.00.

Kito, Fumiki. On virtual mass of a grating of flat plates vibrating in water. Proc. Fac. Eng. Keio Univ. 5, 19–25 (1952).

The virtual mass is calculated by mapping the grating on a circle. The author shows that superimposed potential flow does not affect the virtual mass.

L. M. Milne-Thomson (Greenwich).

Kito, Fumiki. On vibration of a cylindrical shell, which is filled with water. Proc. Fac. Eng. Keio Univ. 4, 95–100 (1951).

Calculation of the virtual mass of a vibrating circular cylindrical shell filled with incompressible inviscid fluid, the ends being closed by rigid plates.

L. M. Milne-Thomson.

Kravtchenko, Julien. Additions à la "Note sur les solutions approchées du problème des sillages". Ann. Inst. Fourier Grenoble 4 (1952), 141–143 (1954).

See same Ann. 3, 287–299 (1952); these Rev. 14, 325.

***Nekrasov, A. I. Točnaya teoriya voln ustanovivšegocya vida na poverhnosti tyaželoĭ židkosti. [The exact theory of steady waves on the surface of a heavy fluid.]** Izdat. Akad. Nauk SSSR, Moscow, 1951. 94 pp. 4 rubles.

This booklet is devoted to an exposition of the author's proof of the existence of periodic two-dimensional gravity waves satisfying the exact boundary condition on the free surface and to his treatment of a nonlinear integral equation. The author's work was prior to Levi-Cività's [Math. Ann. 93, 264–314 (1925)] and independent of Struik's [ibid. 95, 595–634 (1926)]; however, it appeared in publications with very small distribution outside the USSR [Izv. Ivanovo-Vosnosensk. Politehn. Inst. 3, 52–65 (1921); 6, 155–171 (1922); Trudy Vseross. S"ezda Matematikov, Moscow, 1928, pp. 258–262] and consequently has not been well known.

Chap. I considers the case of infinitely deep water, and Chap. II the case with a horizontal bottom. We describe the second case briefly. By means of conformal mappings similar to those used by Levi-Cività the author reduces the problem to that of solving the nonlinear integral equation

$$(1) \qquad \Phi(\theta) = \mu \int_0^{2\pi} \frac{\sin \Phi(\epsilon)}{1+\mu\int_0^{\epsilon} \sin \Phi(\omega) d\omega} K(\epsilon, \theta) d\epsilon,$$

where

$$K(\epsilon,\theta)=\sum_{n=1}^{\infty}\frac{1}{3n\coth(2\pi nh/\lambda)}\cdot\frac{\sin n\theta\cdot\sin n\epsilon}{\pi}.$$

Here λ is the wave length, h the mean depth, Φ is the slope of the free surface as a function of an auxiliary variable θ, and μ is a parameter. For the case of waves of very small slope the author linearizes equation (1) to:

$$(2)\qquad \Phi(\theta)=\mu\int_0^{2\pi}\Phi(\epsilon)K(\epsilon,\theta)d\epsilon.$$

This has for $\mu=\mu_1=3\coth(2\pi h/\lambda)$ the solution $\Phi=\alpha\sin\theta$; however, the associated wave profile and velocity do not coincide with the results of the conventional linearized theory. For the nonlinearized theory, the author looks for solutions of (1) in the form $\Phi=\sum_1^\infty(\mu-\mu_1)^k\Phi_k(\theta)$. The Φ_k can then be found recursively. Φ finally appears as a series in $\sin n\theta$ with coefficients polynomials in $(\mu-\mu_1)$. Convergence is established for sufficiently small $|\mu-\mu_1|$ by using a somewhat different method of procedure and finding majorizing functions for which convergence can be established (not all details were clear to the reviewer).

The final chapter discusses nonlinear integral equations of the form:

$$f(x)=\mu\int_a^b\{f(y)+\epsilon R[\mu,y,f(y)]\}K(x,y)dy,$$

where $K(x,y)=\sum_1^\infty\varphi_i(x)\varphi_i(y)/\mu_i$, $\{\varphi_i\}$ orthonormal over $[a,b]$. The method again is expansion in powers of $(\mu-\mu_1)$ and recursive determination of the terms.

J. V. Wehausen (Providence, R. I.).

Stoker, J. J. Unsteady waves on a running stream. Comm. Pure Appl. Math. 6, 471–481 (1953).

Lorsque l'on perturbe un écoulement uniforme et horizontal d'un liquide pesant, par exemple par immersion d'un obstacle ou par application d'une pression supplémentaire sur une portion de la ligne libre, il s'établit généralement un nouveau mouvement permanent du fluide, qui n'est plus uniforme. En utilisant la théorie linéarisée du mouvement à potentiel d'un fluide pesant avec surface libre l'auteur étudie l'existence et l'allure à l'infini de l'écoulement perturbé ainsi crée. Il est supposé que la perturbation est causée par l'introduction brusque d'une pression constante sur un morceau de ligne libre, et le mouvement permanent est obtenu comme limite, pour un temps infini, de la solution d'un problème aux conditions initiales.

L'utilisation de la transformation de Fourier permet d'expliciter cette solution sous forme d'une intégrale portant sur une fonction dépendant des données initiales, et dont l'étude des singularités fournit le comportement du mouvement quand le temps devient très grand. Le rôle du paramètre gh/U^2 est mis en évidence (h et U désignant la profondeur et la vitesse de l'écoulement non troublé). Si $gh/U^2<1$ le mouvement permanent limite existe et celui-ci est uniforme aux deux infinis; si $gh/U^2>1$ il existe encore un mouvement permanent mais la perturbation s'amortit alors seulement en amont, le profil libre ayant une allure sinusoïdale en aval. Enfin quand $gh/U^2=1$ le problème posé n'admet pas de solution; pour cette valeur critique du paramètre la théorie linéarisée ne permet plus l'étude des phénomènes.

R. Gerber (Toulon).

Chandrasekhar, S. The stability of viscous flow between rotating cylinders in the presence of a radial temperature gradient. J. Rational Mech. Anal. 3, 181–207 (1954).

The equations governing two-dimensional disturbances of a viscous flow between rotating cylinders in the presence of a radial temperature gradient are derived. It is then assumed that the two cylinders are rotating at the same constant angular velocity and the equations governing marginal stability are obtained. Two cases are considered. In Case I, the temperature gradient is proportional to the radius. The resultant characteristic-value problem for $S_n=\Omega^2\alpha\beta(\kappa\nu)^{-1}R_2^6$ is then formulated as a variational problem. In the above formula α denotes the coefficient of volume expansions, β is the constant of proportionality for the temperature gradient, ν and κ are the coefficients of kinematic and thermometric conductivity, and R_2 is the radius of the outer cylinder, and n denotes a pattern of fluid motion with $2n$ vortices. Values of S_n are computed for different values of η, where η denotes the ratio of the radii of the inner to the outer cylinder. It is found that the pattern of connection which first appears at marginal stability shifts progressively to systems with a larger number of vortices as η approaches unity. In Case II it is assumed that the temperature gradient is proportional to the reciprocal of the radius, i.e. the confining cylinders are maintained at constant but, different, temperatures. In this case, the resultant eigenvalue problem for $S_n=\Omega^2\alpha\beta(\kappa\nu)^{-1}R_2^4$ can not be formulated as a variational problem. A method similar to the Galerkin method is used to obtain approximate values of S_n. The variation of S_n with η is the same as in Case I.

R. C. DiPrima.

***Prandtl, Ludwig. Turbulenz.** Naturforschung und Medizin in Deutschland, 1939–1946, Band 11. Hydro- und Aerodynamik, pp. 55–78. Verlag Chemie, Weinheim, 1953. DM 14.00.

***Betz, Albert. Kompressible Strömungen.** Naturforschung und Medizin in Deutschland, 1939–1946, Band 11. Hydro- und Aerodynamik, pp. 79–95. Verlag Chemie, Weinheim, 1953. DM 14.00.

Imai, Isao. Some particular solutions of compressible flow equations for arbitrary pressure-density relation. J. Phys. Soc. Japan 8, 799–801 (1953).

Dans un travail récent Krzywoblocki [même J. 8, 387–389 (1953); ces Rev. 15, 73] a considéré une solution particulière des équations de l'hodographe d'un fluide compressible qui correspond à un écoulement adiabatique dû à un tourbillon. Ce problème avait été déjà traité comme un cas particulier par Ringleb dans sa théorie de l'hodographe [Z. Angew. Math. Mech. 20, 185–198 (1940); ces Rev. 2, 169] et Krzywoblocki a indiqué que son résultat différait de celui de Ringleb.

Dans cette courte note, l'auteur fait remarquer que les résultats précédents sont inclus dans ceux qu'il a établis dans un travail antérieur sur les fluides compressibles avec une loi de compressibilité arbitraire [Appl. Math. Mech. (Ōyō Sûgaku Rikigaku) 1, 147–161 (1947)], et il montre que la différence entre les résultats de Ringleb et de Krzywoblocki tient à ce que le premier auteur a supposé que le coefficient adiabatique γ était égal à 1.4.

R. Gerber (Toulon).

Manwell, A. R. The variation of compressible flows. Quart. J. Mech. Appl. Math. 7, 40–50 (1954).

With a view towards explaining the breakdown of transonic flow, the author considers the variation of a steady

plane potential flow of a compressible fluid. His basic result is that perturbations in the physical plane of the potential ϕ and the stream function ψ satisfy some particularly simple equations in the hodograph variables, namely $\rho q\chi_q+\eta_\theta=0$, $q\eta_q-(\rho q)_q\chi_\theta=0$, where $\chi=\delta\phi$, $\eta=\delta\psi$. He observes furthermore that these perturbations, although obviously distinct from ordinary hodograph solutions, can be expressed in terms of them in an elementary way. As an application, a transonic vortex flow outside a circular cylinder is proved unstable, since there are always (analytically simple) perturbations of the boundary either causing proportionately large changes in the flow or for which no corresponding flow perturbations even exist. (The explanation of this behaviour, analogous to resonance in vibration theory, is that there are critical values of the boundary speed for which the Dirichlet problem for η is non-unique, and these values are dense in the supersonic range.) This example of instability seems to support Frankl's argument, that a transonic flow is the product of an overdetermined boundary-value problem and hence subject to breakdown upon the slightest alteration of conditions [Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 199–202 (1947) = NACA Tech. Memo. no. **1251** (1950); these Rev. **11**, 753]. *J. B. Serrin* (Cambridge, Mass.).

Stanyukovič, K. P. A new approximate method of integration of some equations of hyperbolic type. Doklady Akad. Nauk SSSR (N.S.) **93**, 979–982 (1953). (Russian)

The author applies to the equations of one-dimensional time-dependent compressible flow in Lagrangian form a device previously used by S. A. Christianovitch [Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 215–222 (1947) = Amer. Math. Soc. Translation no. **10**, 3–16 (1950); these Rev. **9**, 390; **11**, 272] to obtain approximate solutions of the equations of steady plane flow. Let $h=\int_0^{x_0}\rho_0 dx_0$ be the mass of the gas column of unit cross-sectional area between 0 and x_0 at time $t=0$. Approximate $pv^k=\exp[(S-S_0)/c_v]$ by $p=B(S)-A^2(S)v$ where $\rho v=1$. Note that $S=S(h)$ and define τ by $A(S(h))d\tau/dh=1$. Then the particle velocity satisfies (i) $A\partial^2u/\partial t^2=\partial(A\partial u/\partial\tau)/\partial\tau$. If $A(S(h))$ is chosen to consist of arcs $A_0(\tau+\tau_0)^{2n}$ where A_0, τ_0, and n are constants, then (i) becomes the Euler-Darboux equation $\partial^2u/\partial t^2=\partial^2u/\partial\tau^2+2n\tau^{-1}\partial u/\partial\tau$. If n is a positive or negative integer or zero the general solution can be expressed in terms of n-fold integrals or derivatives of two arbitrary functions of $t\pm\tau$, and general forms can also be found for v and p. Similarly, the approximation

$$r^N\partial p/\partial h=B(S)-\partial(A^2(S)v/r^N)/\partial h$$

again leads to (i) for cylindrically ($N=1$) and spherically ($N=2$) symmetrical flows, to be followed by the same approximation for $A(S(h))$. *J. H. Giese.*

Riabouchinsky, Dimitri. Application de la méthode des variables topographiques à l'étude des mouvements fluides non-permanents. C. R. Acad. Sci. Paris **238**, 636–638 (1954).

Rappel de quelques applications de la méthode des variables topographiques aux mouvements permanents des fluides compressibles [Riabouchinsky, Publ. Sci. Tech. Ministère de l'Air, Paris, no. **157** (1939)]. Nouvelle extension de cette méthode à des mouvements non-permanents. *From the author's summary.*

Tricomi, Francesco G. Correnti fluide transoniche ed equazioni a derivate parziali di tipo misto. Univ. e Politecnico Torino. Rend. Sem. Mat. **12**, 37–52 (1953).

Cet article est la publication d'une conférence. L'auteur, universellement connu actuellement pour son mémoire sur les équations du type mixte [Atti Acad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (5) **14**, 133–247 (1923)], expose comment l'équation qu'il a introduite (la fameuse équation de Tricomi) s'est trouvée avoir d'importantes applications pour l'étude des écoulements stationnaires à deux dimensions. Partant des équations générales de la mécanique des fluides, il expose la classique méthode de l'hodographe et montre comment choisir la loi d'état du fluide pour réduire l'étude des problèmes transsoniques à celle des solutions d'une équation de Tricomi. Aucun résultat n'est éssentiellement nouveau dans cet article mais l'exposé est particulièrement clair. *P. Germain* (Paris).

Stern, Marvin. On a result of Nikolskii and Taganov concerning transonic flow. J. Aeronaut. Sci. **21**, 133–134 (1954).

The author attempts to give a simple proof that no streamline in a locally supersonic region of irrotational plane flow can contain a straight segment [Nikolsky and Taganov, Akad. Nauk SSSR. Prikl. Mat. Meh. **10**, 481–502 (1946) = NACA Tech. Memo. no. **1213** (1949); these Rev. **8**, 237; **10**, 639]. His derivation of the fundamental lemma that the inclination of the velocity is strictly decreasing along the sonic line is incomplete. It fails to consider the possibility that the supersonic region may contain a simple wave terminated at a straight segment of the sonic line. *J. H. Giese* (Havre de Grace, Md.).

Birkhoff, Garrett, and Walsh, John M. Note on maximum shock deflection. Quart. Appl. Math. **12**, 83–86 (1954).

In this paper a formula is derived, in a somewhat simpler way than usual, for the maximum deflection by a shock wave of a supersonic stream of given Mach number. *D. C. Pack* (Glasgow).

***Teofilato, Pietro. Analogia e similitudine idrogasdinamica.** Società Italiana per il Progresso delle Scienze, XLII riunione, Roma, 1949, Relazioni, Vol. primo, pp. 167–175. Società Italiana per il Progresso delle Scienze, Roma, 1951.

An analogy between steady two-dimensional flows in shallow water and in a gas with specific heat ratio $k=2$ has been discussed by von Kármán [Z. Angew. Math. Mech. **18**, 49–56 (1938)]. The author sketches his own work, published elsewhere in detail, on establishing a correspondence between plane gas flows with $k_0=1.4$ and $k=2$ to make it possible to interpret water table results for air. For irrotational flows [Monografie Sci. Ministero Aeronaut. **1947**, no. 5] this is done by identifying Legendre transforms of the velocity potentials under a mapping which in supersonic portions of the flows takes hodograph characteristics for k_0 onto those for k. To obtain simpler explicit results an alternative linearized discussion is given. The latter method has also been applied to axisymmetric and rotational plane flows [ibid. **1949**, no. 10; Proc. 7th Internat. Congress Appl. Mech., 1948, v. 4, pp. 19–27; these Rev. **11**, 474]. *J. H. Giese* (Havre de Grace, Md.).

Ray, M. Variation of temperature due to small steady disturbances in a compressible flow. Bull. Calcutta Math. Soc. **45**, 45–49 (1953).

The author considers steady, small fluctuations of the single velocity component u superimposed upon a uniform stream $(u=U)$ of viscous, heat-conducting fluid. The fluctuations are required to satisfy some very particular conditions. The general form obtained for the velocity fluctuations does not satisfy the conditions imposed upon the derivatives of these fluctuations. *D. C. Pack.*

✶Küssner, Hans Georg, und Billing, Heinz. Instationäre Strömungen. Naturforschung und Medizin in Deutschland, 1939–1946, Band 11. Hydro- und Aerodynamik, pp. 127–180. Verlag Chemie, Weinheim, 1953. DM 14.00.

This summary of German work in non-stationary flow during 1939–1946 is subdivided as follows. Theoretische Grundlagen (von H. G. Küssner): Methoden der linearisierten Strömungsphysik; Instationäre Tragflächentheorie; Meteorologie; Stabilitätsprobleme und pulsierende Strömungen; Ausbreitung radialsymmetrischer Wellen und Verdichtungsstösse in idealen Gasen; Berichtigungen einiger oben zitierter Arbeiten.

Flugelflattern und verwandte Erscheinungen (von H. G. Küssner): Einleitung; Grundlagen der Flattertheorie; Mathematische Hilfsmittel; Spezielle theoretische Untersuchungen; Experimentelle Untersuchungen; Statistische und dimensionalanalytische Untersuchungen.

Akustik bewegter Schallquellen und -empfänger (von H. Billing): Geradlinig bewegte Schallquellen, Anströmgeräusche; Der Propeller als rotierende Schallquelle; Geradlinig bewegte Schallempfanger, verschiedene Empfängertypen; Schallstörungen durch den Träger der bewegten Schallempfänger.

The material of the first two parts of the first section follows closely Küssner's work [Luftfahrtforschung **17**, 370–378 (1940) = NACA Tech. Memo. no. **979** (1941); these Rev. **2**, 330, 331]. Also included in the second part are references to and notes on the work of Wielandt, Dietze, and Schade on Possio's integral equation, of v. Borbely and Schwarz on the two-dimensional supersonic problem, and of Kinner, Krienes, and Schade on the circular wing in incompressible flow. The third part notes work of Lyra, Stümke, Görtler, Prandtl, Rothstein and Thiriot. The fourth notes Görtler's work on boundary-layer growth and stability, Kaden and Anton's work on vortex formation at the edge of a plate, and Shultz-Grunow's work on pulsating flow in tubes. The fifth deals with the work of Guderley, Bechert, Sauer, Pfriem, and Schultz-Grunow on time-dependent problems in gas dynamics involving only one space coordinate.

The first two parts of the second section are concerned with the formulation of the flutter problem, giving particular reference to work of Teichmann and Küssner. The third notes various sources of numerical results for the aerodynamic forces on two-dimensional airfoils in subsonic, compressible flow and discusses possible, mathematical criteria of flutter.

The third section is concerned primarily with the work of Billing and Merbt on the sound field of a propeller, including the effects of forward flight. *J. W. Miles.*

de Jager, E. M. The aerodynamic forces and moments on an oscillating aerofoil with control-surface between two parallel walls. Nationaal Luchtvaartlaboratorium, Amsterdam. Report **F. 140**, i+15 pp. (16 plates) (1953).

Nelson, Herbert C., and Berman, Julian H. Calculations on the forces and moments for an oscillating wing-aileron combination in two-dimensional potential flow at sonic speed. NACA Rep. no. **1128** (1953), ii+16 pp. (1954).

"Supersedes NACA Tech. Note no. **2590** (1952)"; these Rev. **13**, 880.

Timman, R. Linearized theory of the oscillating airfoil in compressible subsonic flow. J. Aeronaut. Sci. **21**, 230–236, 250 (1954).

Stanitz, John D. Design of two-dimensional channels with prescribed velocity distributions along the channel walls. NACA Rep. no. **1115**, ii+40 pp. (1953).

Supersedes NACA Tech. Note nos. **2593, 2595** (1952); these Rev. **13**, 699. *D. C. Pack* (Glasgow).

Eckart, Carl. The generation of wind waves on a water surface. J. Appl. Phys. **24**, 1485–1494 (1953).

L'auteur applique sa méthode statistique pour l'étude des milieux continus [Physical Rev. (2) **91**, 784–790 (1953); ces Rev. **15**, 175] au problème de génération des ondes gravifiques de surface. Le vent est supposé être constamment normal à la surface de l'eau. En introduisant un certain nombre d'hypothèses simplificatrices, l'auteur calcule la racine quadratique moyenne de P, valeur de la pression du vent au centre de l'orage. Malheureusement le résultat obtenu est dix fois plus grand que sa valeur réelle. L'auteur essai d'expliquer cette divergence. *M. Kiveliovitch.*

Kitkin, P. A. On the profile of the free surface and of the surfaces of separation in a sea of variable density. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. **1953**, 526–545 (1953). (Russian)

L'auteur étudie la variation de la surface libre de la mer en supposant deux surfaces de disontinuité de densité. Ce cas correspond au cas réel: en été et en automne on observe dans la mer deux couches homogènes séparées par une couche de densité différente. L'auteur calcule les échanges des quantités de mouvement dans les deux directions horizontale et verticale, par suite de la turbulence et de l'introduction de la force de Coriolis. L'auteur étudie ensuite l'influence du vent non uniforme ainsi que les oscillations d'inertie. L'auteur applique la théorie à quelques exemples numériques. *M. Kiveliovitch* (Paris).

Miles, John W. On radiation and scattering from small cylinders. J. Acoust. Soc. Amer. **25**, 1087–1089 (1953).

Rayleigh's work on scattering and radiation of harmonic disturbances by thin cylinders is extended to transient disturbances as suggested by a mathematically analogous problem in supersonic flow [G. N. Ward, Quart. J. Mech. Appl. Math. **2**, 75–97 (1949); these Rev. **10**, 644]. A radiation formula, obtained originally by von Kármán in connection with the drag on a slender body of revolution flying at supersonic speed, is derived. *C. J. Bouwkamp.*

Mintzer, David. Wave propagation in a randomly inhomogeneous medium. II. J. Acoust. Soc. Amer. **25**, 1107–1111 (1953).

L'auteur discute en détails, les résultats de la première partie de cette étude [même J. **25**, 922–927 (1953); ces Rev. **15**, 481], et donne une solution approximative de la deuxième approximation de la pression. *M. Kiveliovitch.*

Zatzkis, Henry. Sound field of a moving cylinder and a moving sphere. J. Acoust. Soc. Amer. **26**, 169–173 (1954).

The author obtains formal solutions to the linearized equation for the velocity potential F subject to the boundary condition $F=F_0e^{i\omega t}$ on either a circular cylinder or a sphere moving with constant velocity. In the opinion of the reviewer, the results are of very little physical interest since: (a) the linearized equations for the motion of the bodies in question are valid only for very small values of the Mach number, in which case considerable simplifications are possible; (b) the boundary condition considered is quite artificial, particularly in avoiding the requirement that the flow be tangent to the body. The author's closing statement, viz. "It would be interesting to find out whether the above method can be used to find the sound field of a source moving with sonic or supersonic speed," is of course answered by (a) above, at least if the source is a cylinder or sphere. *J. W. Miles* (Los Angeles, Calif.).

Hönl, H. On the sound field of a point-shaped sound source in uniform translatory motion. NACA Tech. Memo. no. **1362**, 44 pp. (1954).

Translated from Ann. Physik (5) **43**, 437–464 (1943); these Rev. **8**, 113.

Elasticity, Plasticity

***Schmidt, Kurt. Allgemeine Elastizitätstheorie.** Naturforschung und Medizin in Deutschland, 1939–1946, Band 4. Angewandte Mathematik, Teil II, pp. 7–12. Verlag Chemie, Weinheim, 1953. DM 10.00.

Čurikov, F. S. On a form of general solution of the equations of equilibrium of the theory of elasticity in displacements. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 751–754 (1953). (Russian)

The author obtains a general solution to the equations of linear elasticity for isotropic materials. When the body force vanishes, the displacement vector is expressed in terms of a harmonic scalar and a harmonic vector. The author makes a statement to the effect that his solution is not obviously equivalent to that obtained by Galerkin [C. R. Acad. Sci. Paris, **190**, 1047–1048 (1930)]. *J. L. Ericksen.*

Krettner, J. Beitrag zur Anwendung der Tensorrechnung auf die Theorie der Schalen. Ing.-Arch. **21**, 339–345 (1953).

Using vector and tensor methods, the author obtains standard results on the geometry of infinitesimal deformation of shells [cf. Synge and Chien, von Kármán Anniversary Volume, Calif. Inst. Tech., 1941, pp. 103–120; these Rev. **3**, 30]. *C. Truesdell* (Bloomington, Ind.).

Krettner, J. Beitrag zur Berechnung schiefwinkliger Platten. Ing.-Arch. **22**, 47–54 (1954).

Skew (parallelogram) plates under uniform loading and pinned edges are treated by expressing the Lagrange plate equation with independent variables along oblique coordinate axes which are parallel to the edges of the skew plate. The deflection is taken as a double Fourier expression in the new coordinates and satisfies the edge conditions. By a fortunate choice of forms for the coefficients in the series solution, the double sum series for the deflection is reduced to a single sum of trigonometric terms in one independent variable with coefficients containing hyperbolic functions of the other variable. The final deflection function is quite complicated but is shown to degenerate to the known solution of a rectangular plate both in the double sum and single sum form. The author adds an additional solution of the homogeneous equation $\nabla^4 w_1=0$ in order to attack skew plates with other than pinned edges by the superposition of the two solutions. In a forthcoming paper some extensions are to be treated. *D. L. Holl* (Ames, Iowa).

Mitchell, L. H. A Fourier integral solution for the stresses in a semi-infinite strip. Quart. J. Mech. Appl. Math. **7**, 51–56 (1954).

A semi-infinite strip is reinforced by uniform stiffeners along its long edges. Uniform direct stress is applied only to the ends of the stiffeners. This paper treats the elastic diffusion of stress from the stiffeners to the strip. It is assumed that the stiffeners carry direct stress only, and that the strip is in a state of plane stress. The solution is expressed in terms of a stress function given as a Fourier integral. Approximate numerical computations of the shear stress at the edge of the strip are in reasonable agreement with values based upon a photo-elastic investigation.

H. G. Hopkins (Providence, R. I.).

Saroja, B. V. The torsion of solid regular hexagonal shaft by relaxation methods. J. Indian Inst. Sci. Sect. B. **36**, 37–42 (1954).

Ponce, A. Comparison between different methods of calculation applicable to the solution of the problem of torsion of a bar. Bol. Fac. Ingen. Montevideo **5**, 17–40 (1954). (Spanish)

Chandra Das, Sisir. On the effect of a small spherical cavity in a semi-infinite elastic solid under stresses produced by a couple on the plane boundary. Bull. Calcutta Math. Soc. **45**, 89–93 (1953).

Dipolar space coordinates—previously used by Sternberg and Sadowsky [J. Appl. Mech. **19**, 19–27 (1952); these Rev. **14**, 926] in connection with a problem of torsion-free axisymmetry—are applied to a problem of pure torsion. The problem considered is that presented by an elastic halfspace with a spherical cavity under the action of a point-couple applied to the plane boundary, the axis of the couple being coincident with that diameter of the cavity which is normal to the plane boundary. The solution is obtained in form of a series whose coefficients are characterized by a system of recurrence relations. The word "small" in the title of the paper should be omitted. *E. Sternberg.*

Sen, Bibhutibhusan. Note on two-dimensional indentation problems of a non-isotropic semi-infinite elastic medium. Z. Angew. Math. Physik **5**, 83–86 (1954).

Using complex-variable methods, the author obtains a solution to the problem of circular indentation of the straight edge of a semi-infinite plate by a rigid punch, using the equations of linear elasticity for anisotropic materials of a somewhat restricted type. *J. L. Ericksen.*

Eubanks, R. A. Stress concentration due to a hemispherical pit at a free surface. J. Appl. Mech. **21**, 57–62 (1954).

This paper contains a solution in series form for the stresses and displacements around a hemispherical pit at a

free surface of an elastic body. The problem is idealised by considering a semi-infinite medium which is otherwise bounded by a plane. At infinity the body is assumed to be in a state of plane hydrostatic tension perpendicular to the axis of symmetry of the pit. The present method of solution may be generalised to loadings which are not rotationally symmetric. Numerical results are given for the variation along the axis of symmetry of the normal stress which is parallel to the tractions at infinity; these results are compared with the known corresponding numerical values appropriate to the two-dimensional analog of the present problem. (Author's summary.) *R. M. Morris.*

✶**Schmidt, Kurt. Schwingungen elastischer Körper.** Naturforschung und Medizin in Deutschland, 1939–1946, Band 4. Angewandte Mathematik, Teil II, pp. 87–89. Verlag Chemie, Weinheim, 1953. DM 10.00.

Radok, J. R. M. General instability of simply supported rectangular plates. J. Aeronaut. Sci. 21, 109–116 (1954).

The problem of lateral bending of thin plates, stiffened by line reinforcements, is reduced to the limiting case of uniform plates under triangular loading, distributed over a strip, when the width of the strip tends to zero. Using the basic equations, arising from this limiting process, the characteristic equations for the compressive buckling loads of rectangular plates, reinforced by stringers, by ribs, by distributed stringers, and by discrete ribs, are deduced. In all these cases closed expressions in terms of elementary functions are obtained for arbitrary spacing of the reinforcing members. It is shown that the case of discrete stringers and ribs cannot be solved in a similarly simple manner. *R. Gran Olsson* (Trondheim).

Stevens, G. W. H. Corrigendum: The stability of a compressed elastic ring and of a flexible heavy structure spread by a system of elastic rings. Quart. J. Mech. Appl. Math. 7, 128 (1954).

See same J. 5, 221–236 (1952); these Rev. 14, 223.

Vaněk, Jiří. A contribution to the theory of elastic waves produced by shock. Czechoslovak J. Phys. 3, 97–119 (1953). (Russian summary)

Suppose that elastic waves in an infinite, homogeneous and isotropic medium originate at the boundary of a spherical cavity of radius a. The boundary conditions are $p_{rr}=f(t)P_n(\cos\varphi)$, $p_{r\varphi}=\chi(t)dP_n/d\varphi$ for $r=a$, P_n being the Legendre polynomial and $f(t)$ and $\chi(t)$ two arbitrary functions of time. Special attention is paid to $f(t)=\sigma t^\nu e^{-\alpha t}$ (and to $\chi=0$ when a source of explosive character is considered). The method of solution is that of a series expansion used by the Japanese seismic school. The displacement components are obtained in the form of Bromwich-Wagner integrals. The integrands are expressed in terms of Hankel functions of the order $n+\frac{1}{2}$ and P_n. Waves of order $n=0$ and $n=1$ are investigated in more detail. Terms determined by poles of integrands are interpreted as the free waves and those depending on the exciting function as forced waves. The solution obtained shows that in a certain vicinity of the source ($r<3a$ for $n=0$ and $r<19a$ for $n=1$) the amplitude is decreasing as $1/r^k$ where $k>1$. The amplitude also depends on parameters of the exciting function.

W. S. Jardetzky (New York, N. Y.)

Satô, Yasuo. Study on surface waves. IX. Nomogram for the group velocity of Love-waves. Bull. Earthquake Res. Inst. Tokyo 31, 255–260 (1953). (Japanese summary)

Chattarji, P. P. Finite deformation in the interior of the earth. Bull. Calcutta Math. Soc. 45, 113–119 (1953).

The author attempts to determine the deformation of a sphere held strained by the mutual gravitation of its parts, using an empirical law for the density distribution in the deformed sphere and Seth's [Philos. Trans. Roy. Soc. London. Ser. A. 234, 231–264 (1935)] stress-strain relations. He concludes that the radius of the sphere separating the region of expansion from the region of contraction is smaller than that predicted by the linear theory. It should perhaps be noted that several authors, e.g., Signorini [Atti 2° Congresso Un. Mat. Ital., Bologna, 1940, Perrella, Rome, 1942, pp. 56–71; these Rev. 8, 421], have shown that Seth's stress-strain relations cannot be obtained from any strain-energy function, and that the author's solution, together with the law of conservation of mass, implies that the density is nonuniform in the strained and in the unstrained material. *J. L. Ericksen* (Washington, D. C.).

Craggs, J. W. Characteristic surfaces in ideal plasticity in three dimensions. Quart. J. Mech. Appl. Math. 7, 35–39 (1954).

The "characteristic surfaces" discussed in the paper are surfaces of weak discontinuity, to which the term characteristic surfaces is normally reserved, and surfaces of strong discontinuity (singular surfaces). [The fact that the manuscript was received in May 1952 explains the lack of reference to Thomas' comprehensive investigations of these surfaces [J. Rational Mech. Anal. 1, 343–357 (1952); 2, 339–381 (1953); these Rev. 14, 113, 820].] It is shown that the third invariant of the stress deviation vanishes at each point of a characteristic surface. An example of axially symmetric flow with a singular surface is given. *W. Prager.*

Ross, E. W., Jr., and Prager, W. On the theory of the bulge test. Quart. Appl. Math. 12, 86–91 (1954).

In the "bulge test", a uniform circular metal sheet is rigidly held at its edge, and is subjected to uniform transverse pressure sufficient to cause plastic deformation. A theory is given for plastic-rigid material which obeys the yield condition and flow rule of Tresca and which strain-hardens according to a linear law. Simpler equations result than those obtained by R. Hill [Philos. Mag. (7) 41, 1133–1142 (1950); these Rev. 12, 303] in the corresponding theory for von Mises material, and are solved in closed form for deflections up to the onset of instability. No explanation is yet available of the secondary bulge phenomenon.

H. G. Hopkins (Providence, R. I.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Boolsky, R. Optique géométrique axiomatique. Helvetica Phys. Acta 26, 743–754 (1953).

The author starts from the cosine law for imaging a line element sharply, which he seems to take as necessary and sufficient, and derives from this assumption conclusions about approximate image formation based on the deviation of the characteristic function from an idealized function given by the author. The reviewer has certain doubts regarding the validity of the author's conclusions.

M. Herzberger (Rochester, N. Y.).

Colombo, Giuseppe. Sopra una questione di ottica geometrica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 14, 627–631 (1953).

The author proves that there exist no isotropic optical media where all the rays have constant torsion, except when the torsion is zero, i.e., the rays are plane curves. This proof contains the corollary that in dynamics trajectories, belonging to a constant energy-value having constant non-zero torsion, do not exist. *M. Herzberger* (Rochester, N. Y.).

Šachl, Vladimír. The diffraction of electromagnetic waves on an annular disc. Czechoslovak J. Phys. 3, 177–187 (1953). (Russian summary)

Through the use of the vector potential an integral equation is written for the current on a flat annular ring excited by a plane electromagnetic wave at normal incidence. This is reduced to a formulation for a regular function of one variable by the assumption of the usual singularities at the edge and of an azimuthal variation equal to that of the exciting source. The integral equation is not solved but an approximate solution to the problem is given by the method of Schwarzschild [Math. Ann. 55, 177–247 (1902)] using the answers for the disk given by Bouwkamp [Philips Research Rep. 5, 401–422 (1950); these Rev. 12, 774]. To the reviewer this seems a strange procedure indeed for, as the author remarks in his introduction, Schwarzschild's results give good approximations for objects (originally slits) large with respect to wavelength. The Bouwkamp approximation, however, is for disks small with respect to wavelength. A solution for the small annulus should, it appears, be got directly by Bouwkamp's method rather than by the indirect application of Bouwkamp's results for the disk. *W. K. Saunders* (Washington, D. C.).

Jancel, Raymond, et Kahan, Théo. Propagation des ondes électromagnétiques planes dans un plasma homogène (ionosphère). J. Phys. Radium (8) 15, 26–33 (1954).

Continuing earlier work [C. R. Acad. Sci. Paris 236, 788–798, 1478–1481, 2045–2047 (1953); J. Phys. Radium 14, 533–540 (1953)] the authors study plane-wave propagation in a weakly ionized medium in the presence of a constant magnetic field. A comparison with the theories of Appleton and Hartree is included, as well as a discussion of the approximations in applications of the theory to the ionosphere. *C. J. Bouwkamp* (Eindhoven).

Florian, H. Zur Abstrahlung vom offenen Ende einer Lecherleitung und eines Hohlrohres in grosser Entfernung von der Öffnung. Acta Physica Austriaca 8, 42–62 (1953).

Employing a vector equivalent of the Huygens-Kirchhoff principle, the author reports in detail the calculation of the radiation patterns of an open-ended coaxial cable and a waveguide (circular and rectangular cross-sections). *C. J. Bouwkamp* (Eindhoven).

Caprioli, Luigi. Sul comportamento dei modi TEM nei cavi coassiali in presenza di lieve eterogeneità del dielettrico. Rend. Sem. Mat. Univ. Padova 22, 354–365 (1953).

The author studies the propagation characteristics of a coaxial wave guide filled with a dielectric which is "slowly varying" in the direction of the generator of the guide. *A. E. Heins* (Copenhagen).

Caprioli, Luigi. Onde elettromagnetiche trasversali dei tipi TE, TM nelle guide d'onda rettilinee, con dielettrico eterogeneo. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 86, 291–307 (1952).

The author studies the conditions under which a cylindrical wave guide filled with a non-homogeneous dielectric material (uniform in the direction of the generator of the cylinder) may propagate a TE or TM mode. *A. E. Heins* (Copenhagen).

Jessel, Maurice. A propos des vibrations propres d'une cavité ouverte. C. R. Acad. Sci. Paris 238, 1205–1206 (1954).

The author proposes the combined use of short-circuit and open-circuit vector mode functions and their static complements in the solution of diffraction problems arising in the theory of open cavity resonators. *C. H. Papas*.

Finzi, Bruno. Sul principio della minima azione e sulle equazioni elettromagnetiche che se ne deducono. I, II. Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. (8) 12, 378–382, 477–480 (1952).

These two papers extend somewhat the theory developed in a paper under the same title previously reviewed [Atti 4° Congresso Un. Mat. Ital., Taormina, 1951, v. 2, Perrella, Roma, 1953; these Rev. 15, 185]. For electromagnetism in vacuo the elements are three vectors j_α, Φ_α, Ψ_α and the tensor

$$(*)\qquad F_{\alpha\beta}=\Phi_{\beta|\alpha}-\Phi_{\alpha|\beta}+\epsilon_{\alpha\beta\gamma\delta}\Psi^{\gamma|\delta};$$

from these elements the following quadratic invariants can be formed:

$$a=F_{\alpha\beta}F^{\alpha\beta},\quad b=\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta},\quad c=j_\alpha\Phi^\alpha,\quad d=j_\alpha\Psi^\alpha,$$
$$e=j_\alpha j^\alpha,\quad f=\Phi_\alpha\Phi^\alpha,\quad g=\Psi_\alpha\Psi^\alpha,\quad h=\Phi_\alpha\Psi^\alpha.$$

The author takes the action-integrand

$$(**)\qquad l=\frac{a}{4}+\frac{\kappa}{4}b-c-\chi d-\frac{1}{2\lambda^2}(f+\mu g+\nu h),$$

where κ, χ, λ, μ, ν are universal constants, λ being a length and the others pure numbers, and applies the principle of stationary action $\delta\int l d\omega=0$ ($d\omega$=4-element of flat space-time) to obtain field equations. Putting $\kappa=\chi=\mu=\nu=0$, he gets a Maxwell field if $1/\lambda=0$ and a Proca-Yukawa field if $1/\lambda\neq0$. Proceeding more generally with two skew symmetric tensors $F_{\alpha\beta}$, $f_{\alpha\beta}$ as in Mie's theory, the former expressed as in (*) and the latter similarly in terms of Φ_α, Ψ_α, he uses the variational equation $\int\delta l d\omega=0$, where

$$\delta l=\tfrac{1}{2}\delta(f_{\alpha\beta}F^{\alpha\beta})+\frac{\kappa}{2}\delta(\epsilon^{\alpha\beta\gamma\delta}f_{\alpha\beta}F_{\gamma\delta})-j_\alpha\delta(\Phi^\alpha+\chi\Psi^\alpha+\chi'\phi^\alpha+\chi''\psi^\alpha)$$
$$-\frac{1}{\lambda^2}(\phi_\alpha+\nu\psi_\alpha+\nu'\Phi_\alpha+\nu''\Psi_\alpha)\delta(\phi^\alpha+\mu\psi^\alpha+\mu'\Phi^\alpha+\mu''\Psi^\alpha).$$

On putting all the constants here occurring equal to zero except $1/\lambda$ and $\nu'(=-1)$, he obtains the field equations

$$\operatorname{div} f_{\alpha\beta}=j_\alpha,\quad \operatorname{rot} f_{\alpha\beta}=0,\quad \operatorname{div} F_{\alpha\beta}=\frac{1}{\lambda^2}(\phi_\alpha-\Phi_\alpha),\quad \operatorname{rot} F_{\alpha\beta}=0.$$

Thus $f_{\alpha\beta}$ is a Maxwell field and $U_{\alpha\beta}=f_{\alpha\beta}-F_{\alpha\beta}$ a Yukawa field; hence $F_{\alpha\beta}$, being the sum of a Maxwell field and a Yukawa field, is a field of Bopp [Ann. Physik (5) 38, 345–384 (1940); these Rev. 2, 336]. *J. L. Synge* (Dublin).

Finzi, Bruno. Sopra una estensione dei campi elettromagnetici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 13, 211–215 (1952).

Commenting on the formula (*) [see preceding review], the author remarks that in general a skew-symmetric tensor $F_{\alpha\beta}$ requires two vectors Φ_α, Ψ_α for its expression in this form; but if $F_{\alpha\beta}$ is irrotational, Φ_α suffices; if $F_{\alpha\beta}$ is solenoidal, Ψ_α suffices; and if $F_{\alpha\beta}$ is harmonic (i.e. irrotational and solenoidal), then we can use either Φ_α (potential) or Ψ_α (antipotential). Taking Φ_α, Ψ_α to be solenoidal ($\Phi_\alpha{}^{|\alpha}=0$, $\Psi_\alpha{}^{|\alpha}=0$), he uses the field equations derived from (**) [see preceding review] to solve for Φ_α, Ψ_α in terms of the derivatives of $F_{\alpha\beta}$, and obtains for the potentials the equations

$$(***)\qquad 2\Box\Psi_\rho+\alpha j_\rho+\lambda^{-2}(\beta\Phi_\rho+\gamma\Psi_\rho)=0,$$
$$\Box\Phi_\rho+(1+\kappa\alpha)j_\rho+\lambda^{-2}[(1+\kappa\beta)\Phi_\rho+(\tfrac{1}{2}\nu+\kappa\gamma)\Psi_\rho]=0,$$

where α, β, γ are certain combinations of κ, χ, μ, ν.

J. L. Synge (Dublin).

Udeschini, Paolo. Sopra un campo estendente quello elettromagnetico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 13, 246–253 (1952).

The author translates into space and time the theory of Finzi (see preceding reviews), taking some physically typical cases in order to find out something about the five disposable constants κ, χ, λ, μ, ν in Finzi's theory. Translating $F_{\alpha\beta}$ into **E** and **H** in the usual manner, Φ_α into a scalar potential V and a vector potential **A**, and Ψ_α similarly into V^* and $\mathbf{A}^*$, he finds that in an electrostatic field only two constants survive (μ, $1/\lambda$), and the field equations become (ρ=electric density)

$$\operatorname{rot}\mathbf{E}=\mu\lambda^{-2}\mathbf{A}^*,\quad \operatorname{div}\mathbf{E}=\rho+\lambda^{-2}V,$$

and hence

$$\Delta V=\rho+\lambda^{-2}V,\quad \Delta\mathbf{A}^*=\mu\lambda^{-2}\mathbf{A}^*.$$

For a magnetostatic field the same two constants survive. For a point charge at rest he obtains the electrostatic field of Yukawa, modified by an additional electric vector orthogonal to the central field. *J. L. Synge* (Dublin).

Storchi, Edoardo. Determinazione dei potenziali nei campi elettromagnetici generalizzati. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 14, 261–267 (1953).

Analysing the work of Finzi and Udeschini [see preceding reviews] based on (**), the author notes that if $\kappa=\chi=\nu=0$, then the general equations (***) for the potentials become

$$\Box\Phi_\alpha=-\lambda^{-2}\Phi_\alpha-j_\alpha,\quad \Box\Psi_\alpha=-\mu\lambda^{-2}\Psi_\alpha,$$

in which the variables are separated (equation of meson field, Proca-Klein-Gordon equation). He asks: Leaving aside physical limitations, what is the true nature of the analytic problem of determining the potentials? His answer is that in the general case the equations (***) separate into retarded-potential equations and meson-field equations, but that this is not true in a special case which must be picked out and studied. This singular case occurs when there is a certain algebraic relation between the basic constants, and then the equations for the potentials read

$$\bar{\Box}U_\alpha=\omega_\alpha,\quad \bar{\Box}\bar{\Box}\Psi_\alpha=\chi_\alpha,$$

where $\bar{\Box}=\Box+\tau$, τ being a combination of the basic constants, and ω_α, χ_α expressible in terms of the current vector j_α and vanishing in the absence of electric charge; U_α is a linear combination of Φ_α and Ψ_α. Calling a function bimesonic if it satisfies $\bar{\Box}\bar{\Box}\phi=0$, he concludes that in the absence of electric charge one of the two potentials of the singular field of Finzi is a bimesonic function if $1+\mu+2\kappa\nu\neq0$ and a biharmonic function in space-time if $1+\mu+2\kappa\nu=0$. The paper ends with some references to methods of solving the equations of the retarded potential and of the meson field. *J. L. Synge* (Dublin).

Lampariello, G. Das elektrische Erdfeld und das magnetische Sonnenfeld; ein Versuch ihre Beziehung auf Grund der relativistischen Elektrodynamik zu erklären. Z. Angew. Math. Mech. 33, 275–279 (1953).

The author proposes the theory that the earth's electric field is generated by electromagnetic induction during the earth's motion in the sun's magnetic field. *A. E. Schild.*

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The author discusses boundary conditions for electromagnetic field variables at the surfaces of moving bodies.

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Durand, Emile. Le champ $\vec{H}$ et l'induction $\vec{B}$ d'un courant linéaire dans le vide. C. R. Acad. Sci. Paris 238, 1394–1396 (1954).

The relationship in vacuo between the magnetic induction **B** and the magnetic field intensity **H** is found in general not to be one of simple proportionality. It is shown that for a closed loop C carrying a current I and a surface S having the loop for its boundary $\mathbf{B}=\mu_0\mathbf{H}+\mu_0 I\delta(\mathbf{n})$ with $\delta(\mathbf{n})=-(4\pi)^{-1}\int_S\mathbf{n}\Delta(1/r)dS$, **n** being the unit vector along the outward normal to S. In the special case where the loop is a circle around the origin in the (x, y)-plane, $B_z=\mu_0H_z+\mu_0I\delta(z)$, where $\delta(z)$ is the Dirac delta function. Results are also derived showing the difference between the circuitation of **B** and **H**. *J. E. Rosenthal.*

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Reza, F. M. RLC canonic forms. J. Appl. Phys. 25, 297–301 (1954).

A canonic representation of a driving-point impedance is here defined as the series connection of positive multiples of a physical impedance $z(s)$ in parallel with positive multiples of $z^{-1}(s)$; or the series connection of positive multiples of $z(s)$ in parallel with positive constants. Let $Z(s)$ be the impedance of a network formed by any interconnections of $z(s)$ and $z^{-1}(s)$. The author asserts, then, that an RLC canonical structure exists if (a) $z(s)$ has only real poles and zeros and (b) the half segment of the curve $\mathrm{Re}[z(s)]=0$ corresponding to $\mathrm{Im}[s]>0$ does not cross the real axis (and similarly for $\mathrm{Im}[s]<0$). This leads to the result that a driving-point impedance function can be synthesized in a canonical form if and only if $Z(s)=F[f(s)]$, where $F(\lambda)$ represents an LC impedance function and $f(s)$ represents an RC or RL impedance function. Generalizations of a class of canonical impedance functions are discussed and some examples given. *R. Kahal* (Monterey, Calif.).

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